Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH3091-WE01

Title: Dynamical Systems III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 The equations of a dynamical system in polar coordinates (r, θ) are given by

$$\dot{r} = (1 - r^2)\sinh r\,, \qquad \dot{\theta} = r\,.$$

Sketch the phase flow of this system both in Cartesian and polar coordinates and explain your reasoning for drawing that particular flow. Please label all interesting points and regions in this flow and explain your reasoning.

Q2 A linear two-dimensional dynamical system satisfies the equation $\dot{\mathbf{x}} = A\mathbf{x}$, where the matrix A is

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix},$$

and $\mathbf{x}$ is a two-dimensional state vector.

(a) Explicitly find a similarity transformation matrix M, and use it to rewrite A in Jordan normal form. Use this to write the explicit solution for the linear system in the original coordinates $\mathbf{x}$.

(b) Sketch the phase flow for this linear system.

Q3 Consider a dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$, where $\boldsymbol{F}(\boldsymbol{x})$ depends smoothly on $\boldsymbol{x}$ and a parameter μ.

(a) What is the necessary conditions for the system to undergo a local bifurcation at $\boldsymbol{x}^*, \mu^*$? What do these conditions become for one-dimensional systems?

(b) Now suppose the system is one-dimensional, and $F(x) = \sin(x)(\cos(x) - \mu)$, where $-\pi < x < \pi$. Determine the bifurcation points, state the type of bifurcation, and draw the bifurcation diagram at each bifurcation point.

Q4 Consider a two-dimensional dynamical system

$$\dot{x} = f(x, y)\ ,\ \dot{y} = g(x, y)$$

(a) State and prove Bendixson's criterion for the exclusion of periodic solutions to this system in a region $D \subset \mathbb{R}^2$.

(b) Now take

$$f(x, y) = 1 + y - x^2\ ,\ g(x, y) = 3 - 4x + x^2\ . \tag{1}$$

Does this system admit a periodic orbit in the positive quadrant $x, y > 0$? Explain your answer.

(c) Determine the fixed points for the dynamical system (1), and analyse their stability.

(d) Is the dynamical system specified by (1) topologically conjugate to

$$\dot{x} = 1 + y\ ,\ \dot{y} = 3 - 4x\ ?$$

Explain your answer.

SECTION B

Q5 A two-dimensional dynamical system is given by the equations

$$\begin{aligned}\dot{x} &= y\,,\\ \dot{y} &= 4x^3 - x\,.\end{aligned} \tag{2}$$

(a) Work out all fixed points of this system and determine their nature.

(b) Using an educated guess, sketch the phase flow for this system.

(c) Say what homoclinic and heteroclinic orbits are. If any of these exist in this system indicate these orbits in the flow.

(d) State what a Hamiltonian system is and show that the system (2) is Hamiltonian by constructing the Hamiltonian H. Assume that the Hamiltonian is of the form

$$H(x,y) = \alpha(x^2+y^2) + \beta x^4\,.$$

Use the Hamiltonian to work out the equations of the heteroclinic orbits from part (c).

Q6 The autonomous dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{F}(x)$ has function $\boldsymbol{F}(\boldsymbol{x})$ given by

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{pmatrix} -2x + xy + \alpha y \\ 2y - 2\alpha x + x^2 \end{pmatrix}\,.$$

Answer the following questions:

(a) Which properties does the function $\boldsymbol{F}(\boldsymbol{x})$ have to satisfy in order for the stable manifold theorem to be valid for this system? State the stable manifold theorem.

(b) Work out all critical points of this system, and figure out which condition the parameter α has to satisfy so that the stable manifold theorem can be applied.

(c) If the parameter $\alpha = 0$, work out what the stable and unstable submanifolds $m_\pm$ are in the linearised approximation.

(d) Evaluate corrections to the stable and unstable submanifolds in the nonlinear theory. Assume that in the nonlinear theory corrections to the linearised submanifolds have the expansion

$$\begin{aligned} y(x) &= a_x x^2 + b_x x^3 + \ldots \quad (|x| \ll 1)\,,\\ x(y) &= a_y y^2 + b_y y^3 + \ldots \quad (|y| \ll 1)\,. \end{aligned}$$

Substitute these into the equations $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$ to work out the coefficients a_x, a_y and b_x, b_y. Then sketch how these stable and unstable submanifolds look in the linear and nonlinear approximations.

Q7 (a) For the dynamical system

$$\dot{x} = \cos(x) - \cos(y)\ ,\ \dot{y} = \cos(x) + \cos(y) \tag{3}$$

find the Poincaré index for a curve γ that is the boundary of a square with corners $(-3,-3), (3,-3), (3,3), (-3,3)$, and has counter-clockwise direction.

(b) Given the dynamical system (3), what are the ω and α limit sets for the points $(0,0)$, $(\pi/2, \pi/2)$?

(c) Is this a Hamiltonian system? If it is, state the Hamiltonian. If it is not, prove why it is not.

(d) Suppose a two-dimensional dynamical system

$$\dot{x} = f(x,y)\ ,\ \dot{y} = g(x,y)$$

depends smoothly on a parameter μ, and has one fixed point for $\mu < 0$. This fixed point is a saddle point. Is it possible for such a system to undergo a bifurcation, say at $\mu = 0$, to a system with a node, and no other fixed point? Use the properties of the Poincaré index to motivate your answer.

Q8 The two-dimensional dynamical system

$$\dot{x} = -x + ay + x^2 y$$
$$\dot{y} = \frac{1}{2} - ay - x^2 y$$

has one fixed point at

$$(x^*, y^*) = (\frac{1}{2}, \frac{2}{1+4a})\ .$$

We assume $a > 0$.

(a) Determine the stability of the fixed point as a function of the parameter a.

(b) Determine the curve C_1 where $\dot{x} = 0$, and the curve C_2 where $\dot{y} = 0$,

(c) Show that for $x > 1/2$

$$\frac{dy}{dx} < -1$$

along the trajectories of the flow.

(d) Your results from parts (b) and (c) may be used to show that the flow lines of this dynamical system will point inwards on the boundary of a polygon D with vertices in the points

$$(0,0)\ ,\ (0,5)\ ,\ (0.5,5)\ ,\ (5,0.5)\ ,\ (c, 5.5-c)\ .$$

where $c \sim 5.32$, so the last vertex lies below the curve C_1. [You don't have to show this.] Use this observation to argue that D is a compact, positively invariant set of the dynamical system.

(e) Use the Poincare-Bendixson theorem to determine the values of a for which there is a limit cycle in $D \backslash (x^*, y^*)$.

(f) What type of bifurcation does this system exhibit as a is varied?