

EXAMINATION PAPER

Examination Session: May/June

2022

Year:

Exam Code:

MATH3101-WE01

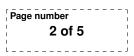
Title:

Fluid Mechanics III

Time:	3 hours	
Additional Material provided:	Formula sheet.	
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.		
	Students must use the mathematics specific answer book.		

Revision:





SECTION A

Q1 (a) By considering conservation of mass in a fluid element D_t moving with the fluid, derive the Eulerian form of the mass continuity equation. You may assume that ρ and **u** and their integrals are well behaved and that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{D_t} f \, dV = \int_{D_t} \left(\frac{\mathrm{D}f}{\mathrm{D}t} + f \nabla \cdot \mathbf{u} \right) \, dV,$$

for any smooth scalar function f.

- (b) Assuming $\mathbf{u} = u(y,t)\mathbf{e}_y$, solve the continuity equation to find u(y,t) when $\rho(y,t) = (2 + \sin(t))y^2$ and u(0,t) = 0.
- Q2 The linearised equations for small-amplitude water waves on the surface of a domain $D_t = \{(x, z) : -h < z < \zeta(x, t), -\infty < x < \infty\}$ are given by

$$\begin{split} \Delta \phi &= 0 \quad \text{in } -h < z < 0, \\ \frac{\partial \phi}{\partial t} + g \zeta &= 0 \quad \text{at } z = 0, \\ \frac{\partial \zeta}{\partial t} &= \frac{\partial \phi}{\partial z} \quad \text{at } z = 0, \\ \frac{\partial \phi}{\partial z} &= 0 \quad \text{at } z = -h. \end{split}$$

- (a) Aside from their small amplitude and the ideal nature of the fluid, what two other main assumptions have been made about the flow in deriving the above equations?
- (b) Assuming a solution of the form $\phi(x, z, t) = X(x)Z(z)\sin(\omega t)$, find the most general solutions for X(x) and Z(z) that satisfy these equations.
- (c) Find the dispersion relation. Are these waves dispersive?
- **Q3** A fixed volume V is filled with an unforced ideal barotropic fluid obeying the relation $P(\rho) = \rho^{\gamma}$ with γ constant.
 - (a) Write down the continuity and momentum equations for this fluid in terms of \boldsymbol{u} and ρ only (not p).
 - (b) Assuming that $\boldsymbol{u} \cdot d\boldsymbol{S} = \boldsymbol{0}$ on the boundary ∂V , use the equations in (a) to show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V} \frac{1}{2} \rho |\boldsymbol{u}|^2 \,\mathrm{d}V = -\int_{V} \gamma \rho^{\gamma-1} \boldsymbol{u} \cdot \nabla \rho \,\mathrm{d}V.$$



- Q4 An incompressible Newtonian viscous fluid is flowing steadily in the x-direction through a pipe of elliptical cross section, with semi-major axis a (in the y-direction) and semi-minor axis b (in the z-direction).
 - (a) Assuming a constant pressure gradient $\frac{dp}{dx} = P$, and a velocity field of the form $\boldsymbol{u} = u(y, z)\boldsymbol{e}_x$, show that the Navier-Stokes equations reduce to the Poisson equation

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{P}{\mu}.$$

- (b) Write down the appropriate boundary condition for u on the surface of the pipe.
- (c) Show that a function of the form $u(y,z) = \alpha \left(\frac{y^2}{a^2} + \frac{z^2}{b^2} 1\right)$ satisfies both the equation and the boundary condition for some constant α which you should find.

SECTION B

- Q5 (a) Write down the Lagrangian form of the vorticity equation for an ideal, incompressible fluid.
 - (b) Starting from the ansatz that vorticity can be expressed in index notation as

$$\omega_i = C_j \frac{\partial x_i}{\partial a_j},$$

where $C(\mathbf{x}, t)$ is some unknown vector field, show that the solution of the Lagrangian form of the vorticity equation can be written as

$$\omega_i(\mathbf{x}, t) = \frac{\partial x_i}{\partial a_j} \omega_j(\mathbf{a}, 0).$$

(c) Consider a fluid flow with vorticity at t = 0 given by

$$\boldsymbol{\omega}(\mathbf{x}, 0) = \begin{cases} \Omega \mathbf{e}_z & x^2 + y^2 \le R_0^2 \\ 0 & x^2 + y^2 > R_0^2, \end{cases}$$

where $\Omega, R_0 \in \mathbb{R}$. Find $\boldsymbol{\omega}(\mathbf{x}, t)$, the vorticity at later times, when this flow is evolved by a potential flow of the form

$$\mathbf{u} = -x\mathbf{e}_x - y\mathbf{e}_y + 2z\mathbf{e}_z.$$

Q6 Consider an ideal fluid flow in the domain between two spheres of radius a and b, where b > a > 0. Working in *cylindrical* coordinates the flow is described by the Stokes' stream function $\psi(r, z)$ such that

$$\mathbf{u}(r,z) = -\frac{1}{r}\frac{\partial\psi}{\partial z}\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial r}\mathbf{e}_z.$$

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(a) Show that the vorticity is given by

$$\boldsymbol{\omega} = -\frac{1}{r} L \boldsymbol{\psi} \mathbf{e}_{\boldsymbol{\theta}},$$

where $L\psi$ is an operator applied to ψ you should determine.

(b) The vorticity between the spheres is such that

$$L\psi = \begin{cases} \Omega r^2 & a \le \rho \le b\\ 0 & \text{otherwise} \end{cases},$$

where $\rho = \sqrt{r^2 + z^2}$ and $\Omega \in \mathbb{R}$. Assuming that the potential takes the form $\psi(r, z) = r^2 f(\rho^2)$, show that in the region $a \le \rho \le b$, f satisfies

$$10f'(t) + 4tf''(t) = \Omega,$$

where $t = \rho^2$, and solve to find the general expression for f(t).

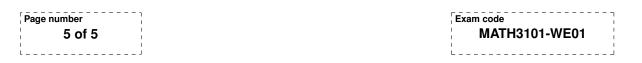
- (c) Assume that $f(\rho = a) = f(\rho = b) = 0$ and find f, expressing your answer in terms of ρ . With these boundary conditions how does the flow behave at $\rho = a$ and b and why?
- **Q7** A bugle is modelled as an infinitely-long straight tube $0 < x < \infty$ with varying cross-sectional area $A(x) = e^{ax}$ for some constant a. We propose to model the air inside the bugle by a one-dimensional flow $\boldsymbol{u} = u(x,t)\boldsymbol{e}_x$, p = p(x,t), $\rho = \rho(x,t)$. For consistency, conservation of mass implies that the continuity equation must be modified to

$$A\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(A\rho u) = 0.$$

- (a) Assuming that they are unmodified, write down the other unforced compressible barotropic Euler equations for this flow.
- (b) Assume a basic state with uniform density ρ_0 , uniform pressure p_0 and zero velocity. Replacing $\rho \to \rho + \rho_0$ and $p \to p + p_0$ in your equations from (a), derive the resulting linearised equations for the perturbations ρ , u and p.
- (c) Show that the pressure perturbation p satisfies the modified wave equation

$$\frac{\partial^2 p}{\partial t^2} = c_0^2 \left(\frac{\partial^2 p}{\partial x^2} + a \frac{\partial p}{\partial x} \right).$$

(d) By taking the ansatz $p(x,t) = X(x)e^{i\omega t}$, find the condition on ω for the existence of travelling waves that move along the tube. How does the amplitude of these waves depend on A?



Q8 An incompressible axisymmetric flow in the infinite cylinder a < r < b has the form

$$u_r(r,z,t)\boldsymbol{e}_r + \left[U(r) + u_\theta(r,z,t)\right]\boldsymbol{e}_\theta + u_z(r,z,t)\boldsymbol{e}_z, \qquad p_0(r) + p(r,z,t), \qquad \rho_0,$$

where u_r , u_{θ} , u_{ϕ} and p represent perturbations around a steady flow $U(r)e_{\theta}$ with pressure $p_0(r)$ and uniform density ρ_0 .

- (a) Write out the four unforced incompressible Euler equations satisfied by u_r , $U + u_{\theta}$, u_z and $p_0 + p$. What boundary conditions are required on r = a and r = b?
- (b) What condition on U and p_0 is required in order that they satisfy the steady Euler equations when there is no perturbation?
- (c) Hence write down the linearised (unforced incompressible Euler) equations satisfied by the perturbations u_r , u_{θ} , u_z , and p.
- (d) By assuming the ansatzes

$$u_r = \hat{u}_r(r)e^{i(kz-\omega t)}, \ u_\theta = \hat{u}_\theta(r)e^{i(kz-\omega t)}, \ u_z = \hat{u}_z(r)e^{i(kz-\omega t)}, \ p = \hat{p}(r)e^{i(kz-\omega t)},$$

with $k \in \mathbb{R}$ and $\omega \in \mathbb{C}$, show that the linearised equations reduce to the single ODE

$$\frac{\mathrm{d}}{\mathrm{d}r} \left[\frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(r \hat{u}_r \right) \right] - k^2 \hat{u}_r = -\frac{k^2}{\omega^2} \Phi(r) \hat{u}_r,$$

where $\Phi(r) = \frac{1}{r^3} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 U^2 \right)$.