Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH3111-WE01

Title:
Quantum Mechanics III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 Consider a Quantum Mechanical system which is described by a two dimensional Hilbert space with orthonormal basis $\{|1\rangle, |2\rangle\}$. The Hamiltonian operator $\hat{H}$ acts on the basis vectors according to,

$$\hat{H}\,|1\rangle = E\,(|1\rangle + i\,|2\rangle)\;,$$
$$\hat{H}\,|2\rangle = E\,(c\,|1\rangle + (1 + i\,b)\,|2\rangle)\;,$$

with b, E real numbers and c complex.

(a) Find the matrix form of $\hat{H}$ and fix the numbers c, b. Find the possible outcomes in a measurement of $\hat{H}$ and its eigenspaces.

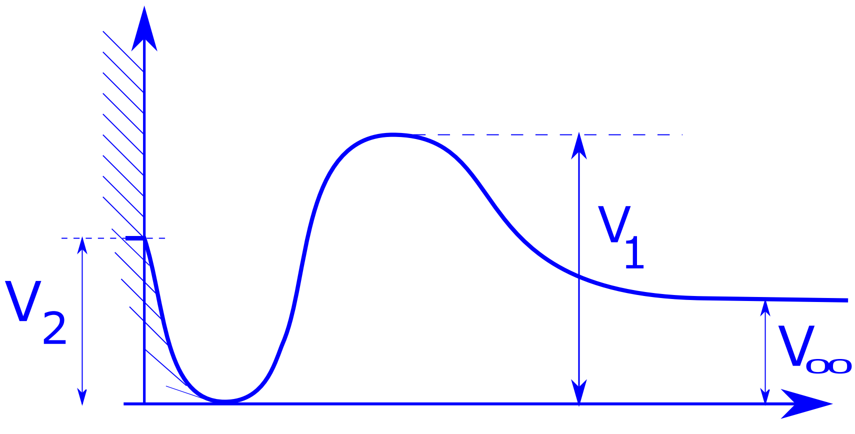
(b) Write the matrix form of the operator $U(t) = \exp\left(-\frac{i\,t}{\hbar}\,\hat{H}\right)$.

Q2 Consider a Quantum Mechanical particle of mass m in one dimension moving inside a potential $V(x)$ which is an analytic function.

(a) Use induction to express the commutator $[\hat{p}, \hat{x}^n]$ for positive integer n, as a power of $\hat{x}$. Use that result to compute the commutator $[\hat{p}, \hat{V}(\hat{x})]$.

(b) Consider a potential of the form $V(x) = \lambda\,x^n$, with λ real and n a positive integer. Express the time derivatives of the expectation values $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$ in terms of the expectation values of powers of $\hat{x}$ and $\hat{p}$.

Q3 A quantum particle is propagating in a one-dimensional potential shown in the image.



In this potential, as shown on the image, $0 < V_\infty < V_2 < V_1$. Answer the following questions:

(a) Is the energy spectrum of the particle in this system discrete or continuous? If your answer depends on the energy of the particle, please explain and state which part of the spectrum is discrete and which is continuous.

(b) Sketch the qualitative form of the wave function for the semi-classical particle in this potential. If the behaviour depends on the energy, discuss all possibilities.

(c) What happens with the energy spectrum if $V_\infty \to 0$?

(d) What happens with the energy spectrum if $V_1 \to \infty$?

Q4 Apply the WKB approximation to compute the energy spectrum of a quantum particle propagating in the potential

$$V(x) = \begin{cases} \infty & x < 0\,, \\ 0 & 0 < x < B\,, \\ e^x & x > B\,. \end{cases}$$

Analyse separately the situations when $E < e^B$ and when $E > e^B$. In the second case you do not need to explicitly solve for the energy of the particle.

Hint: When evaluating one of the integrals, you could solve the integral by replacing the full square root with a new variable.

SECTION B

Q5 Consider the two dimensional simple harmonic oscillator with Hamiltonian,

$$\hat{H} = \hbar\omega \left(\hat{a}_x^\dagger \, \hat{a}_x + \hat{a}_y^\dagger \, \hat{a}_y + 1\right) ,$$

and angular momentum,

$$\hat{L} = i \, \hbar \left(\hat{a}_x \, \hat{a}_y^\dagger - \hat{a}_y \, \hat{a}_x^\dagger\right) .$$

The operators $\hat{a}_x$ and $\hat{a}_y$ satisfy the algebra

$$[\hat{a}_x, \hat{a}_x^\dagger] = [\hat{a}_y, \hat{a}_y^\dagger] = 1 \,,$$

with all the other commutators trivial.

(a) Prove that angular momentum defines a conserved quantity.

(b) Define the operators,

$$\hat{a}_\pm = \frac{1}{\sqrt{2}} \left(\hat{a}_x \pm i \, \hat{a}_y\right) ,$$

and their adjoints $\hat{a}_\pm^\dagger$. Compute all the commutators between $\hat{a}_\pm$ and $\hat{a}_\pm^\dagger$. Express $\hat{H}$ and $\hat{L}$ in terms of them and argue that $\hat{H}$ and $\hat{L}$ should have a common eigenbasis.

(c) Given the ground state $|0\rangle$, consider the general state,

$$|\psi_0\rangle = \sum_{n_+, n_- \geq 0} c_{n_+, n_-} \left(\hat{a}_+^\dagger\right)^{n_+} \left(\hat{a}_-^\dagger\right)^{n_-} |0\rangle \,,$$

with c_{n_+,n_-} complex numbers. Write down the condition which guarantees that $|\psi_0\rangle$ is a unit norm state and express the expectation values $\langle \hat{H} \rangle$ and $\langle \hat{L} \rangle$ in terms of the constants c_{n_+,n_-}.

(d) The state $|\psi'\rangle = \exp\left(i \, \frac{\lambda}{\hbar} \, \hat{L}\right) |\psi_0\rangle$ can be written in the same form as $|\psi_0\rangle$ but with a new set of constants c'_{n_+,n_-} instead of c_{n_+,n_-}. Express the constants c'_{n_+,n_-} in terms of c_{n_+,n_-}.

Q6 The Hilbert space of a Quantum system is spanned by the three orthonormal vectors $\{|1\rangle, |2\rangle, |3\rangle\}$. The Hamiltonian acts on the basis vectors according to,

$$\begin{aligned}\hat{H}|1\rangle &= J\ (c_1\,|1\rangle - i\,|2\rangle + |3\rangle)\ ,\\ \hat{H}|2\rangle &= J\ (c_1\,|2\rangle + i\,|1\rangle - i\,|3\rangle)\ ,\\ \hat{H}|3\rangle &= J\ (|1\rangle + i\,|2\rangle + c_2\,|3\rangle)\ ,\end{aligned}$$

with J, c_1 and c_2 real numbers. The system also contains an operator $\hat{G}$ with its action on the Hilbert space fixed by,

$$\begin{aligned}\hat{G}|1\rangle &= D\ (a\,|1\rangle + i\,|2\rangle)\ ,\\ \hat{G}|2\rangle &= D\ (-i\,|1\rangle + b\,|2\rangle)\ ,\\ \hat{G}|3\rangle &= D\,c\,|3\rangle\ ,\end{aligned}$$

with D, a, b and c real positive numbers.

(a) Write the matrix forms of $\hat{H}$ and $\hat{G}$.

(b) Write the necessary conditions in order for $\hat{G}$ to generate a continuous symmetry of $\hat{H}$.

(c) Write the matrix form of $\hat{g}(\hat{G})$ where $g(x)$ is an analytic function.
Hint: Your final answer should be a matrix which can contain the function g evaluated on specific numbers.

(d) For a state $|\psi\rangle$, the observable $\hat{G}$ is measured yielding the largest value possible. Right after the measurement, the observable $\hat{H}$ is measured. What are the values that can have non-zero probability in the measurement of $\hat{H}$? Consider the case where $c_1 = 2$ and $c_2 = 1$.

Q7 A quantum particle is bouncing inside the empty space between two impenetrable spheres

$$V(x) = \begin{cases} \infty & r < R_1\,, \\ 0 & R_1 < r < R_2\,, \\ \infty & r > R_2\,. \end{cases}$$

The Laplacian in spherical coordinates is given by

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\hat{L}^2}{r^2}\,.$$

Answer the follwing questions:

(a) Write the ansatz for the wave function of this quantum particle, and by separation of variables, work out the radial and angular equations. What are the solutions of the angular equation?

(b) What are the boundary conditions that the wave function has to satisfy?

(c) By changing variables via $u = rR(r)$, simplify the radial equation. What can you say about the spectrum of the particle based on this equation?

(d) By changing variables with $u = rR(r)$, compute the wave function of the ground state and work out its energy.

Q8 A simple harmonic potential in three spatial dimensions has the potential $V(\mathbf{x})$

$$\hat{V}(\hat{\mathbf{x}}) = \kappa\, \hat{\mathbf{x}}^2/2\,.$$

This system is perturbed by adding to the potential $\hat{V}(\hat{\mathbf{x}})$ the term

$$\hat{V}_1' = \alpha(\hat{x}\hat{p}_x + \hat{y}\hat{p}_y + \hat{z}\hat{p}_z).$$

Answer the following questions:

(a) Explicitly write down the spectrum of the unperturbed system, i.e. write down the properly normalized energy eigen-states as well as their energies. What is the degeneracy of the N-th excited level? Please explain your answer.

(b) By explicit computation find the first-order correction to the ground state as well the correction to its energy.

(c) By explicit computation find the corrections to the energies of first and second excited states. What is correction to the energy of the N-th excited state?

(d) What is correction to the ground state and the first excited state, if instead of the perturbation $\hat{V}_1'$ we apply the perturbation $\hat{V}_2' = \alpha\hat{L}_z = \alpha(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)$. You do not need to do any computation here, but you need to explain the answer.

Hint: You may find the following expressions useful:

$$\begin{aligned} \hat{x} &= A(\hat{a}_x + \hat{a}_x^\dagger)\,, \\ \hat{p}_x &= iB(\hat{a}_x - \hat{a}_x^\dagger)\,, \end{aligned}$$

where $A, B > 0$ are constants, and similar expressions hold for y and z. Also ignore the fact that the perturbation operator in the first two parts of the question is not Hermitian.