Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH3141-WE01

Title: Operations Research III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 Starting from the north-west initial assignment, find all optimal basic feasible solutions for the transportation problem with costs, supplies and demands given below:

$$[c_{ij}] = \begin{bmatrix} 3 & 7 & 6 & 4 \\ 2 & 4 & 3 & 2 \\ 4 & 3 & 8 & 5 \end{bmatrix}, \qquad [a_i] = \begin{bmatrix} 50 \\ 20 \\ 30 \end{bmatrix}, \qquad [b_j] = [30\ 30\ 20\ 20].$$

Q2 Write down the general primal/dual pair of linear programming problems in symmetric form and the complementary slackness conditions for this pair of problems. State carefully and prove the weak duality and complementary slackness theorems.

Q3 Electricity demand in a large city follows the following daily cycle (in units of MWh):

time of day	night 0:00-6:00	morning 6:00-12:00	afternoon 12:00-18:00	evening 18:00-0:00
demand	2	4	6	8

City planners would like to quantify the risk of supplying electricity through wind only. We will model energy capacity as a Markov chain with transition probabilities

$$P = \begin{bmatrix} 0.4 & 0.6 & 0 \\ 0.1 & 0.5 & 0.4 \\ 0 & 0.5 & 0.5 \end{bmatrix} \tag{1}$$

State 1 corresponds to a capacity of 3 MWh (over the next 6 hours), state 2 to a capacity of 6 MWh, and state 3 to a capacity of 9 MWh. The chain transitions in intervals of 6 hours, starting at 0:00 (matching the demand table), so there are 4 transitions per day.

If the capacity exceeds demand, then demand is fully met. Otherwise, an energy shortage occurs. For example, if on a specific day from 12am (i.e., 0:00) onward, the Markov chain follows states $(1, 1, 2, 2)$, corresponding to capacities $(3, 3, 6, 6)$, then the daily energy shortage is

$$\max\{2-3, 0\} + \max\{4-3, 0\} + \max\{6-6, 0\} + \max\{8-6, 0\} \\ = 0 + 1 + 0 + 2 = 3 \tag{2}$$

in units of MWh. What is the long-run expected daily energy shortage?

Q4 The operator of a small offshore wind farm needs to decide on how to best maintain their main subsea power cable. If the cable works (state $i = 1$), their monthly income is 20, otherwise (state $i = 2$) it is 0 (in units of £10k). At the end of each month, they need to decide to (with costs expressed in units of £10k):

- do nothing (action $k = 1$) at cost 0,
- pay for cheap cable maintenance (action $k = 2$) at cost 5, or
- pay for expensive cable maintenance (action $k = 3$) at cost 15.

The transition matrix $P(k)$ between monthly states under each action k is:

$$P(1) = \begin{bmatrix} 0.7 & 0.3 \\ 0 & 1 \end{bmatrix} \qquad P(2) = \begin{bmatrix} 0.9 & 0.1 \\ 0 & 1 \end{bmatrix} \qquad P(3) = \begin{bmatrix} 1 & 0 \\ 0.9 & 0.1 \end{bmatrix} \tag{3}$$

The operator is currently paying for cheap maintenance each month if the cable is working, and for expensive maintenance when it is broken. Is this policy optimal with respect to maximizing long-run average profit?

SECTION B

Q5 Consider the following linear programming problem:

$$
\begin{array}{lrcrcrcl}
\text{maximise} & -x_1 & + & x_2 & + & 2x_3 & & \\
\text{subject to} & x_1 & + & 2x_2 & - & x_3 & \leq & 20 \\
 & -2x_1 & + & 4x_2 & + & 2x_3 & \leq & 60 \\
 & 2x_1 & + & 3x_2 & + & x_3 & \leq & 50
\end{array}
$$

and each $x_i \geq 0$.

5.1 Solve this problem using the simplex algorithm.

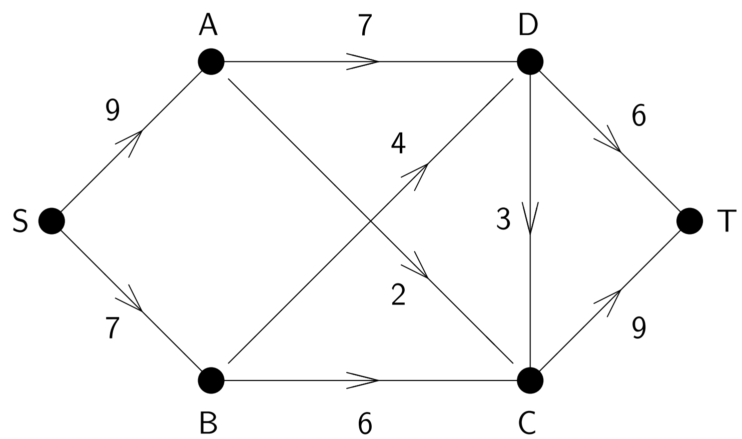
5.2 Let $P(\boldsymbol{\varepsilon})$ be the linear programming problem when the right-hand side $\boldsymbol{b} = (20, 60, 50)^{\mathsf{T}}$ is replaced by the perturbed vector

$$\boldsymbol{b}(\boldsymbol{\varepsilon}) = (20 + \varepsilon_1, 60 + \varepsilon_2, 50 + \varepsilon_3)^{\mathsf{T}}.$$

Give a formula, in terms of $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)^{\mathsf{T}}$, for the optimal value of $P(\boldsymbol{\varepsilon})$ when the ε_i are small. For what set of $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ values does your formula hold?

Q6 **6.1** Describe carefully the maximum flow problem as a linear programming problem. State the result about value of a feasible flow through a network and the minimal cut capacity.

6.2 Use the Ford-Fulkerson algorithm to find a "maximal flow" from the source S to the terminus T for the network below, where the numbers indicate capacities of the corresponding arcs.



Use your solution to determine the corresponding "minimal cut".

Q7 Consider the following optimization problem:

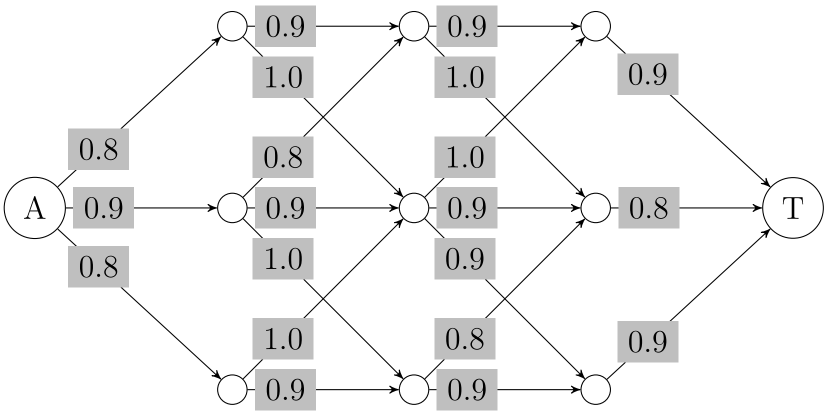
$$\max\ 5x_1 + x_2^2 + x_3^2 \text{ subject to } 2 \le x_1x_2x_3 \le 4 \tag{4}$$

with $x_i \in \{1,2,3,4\}$ for all $i \in \{1,2,3\}$.

7.1 Phrase this problem as a dynamic programming problem.

7.2 Find all solutions to the problem, using dynamic programming.

Q8 Consider the following network, representing guarded routes between hiding spots:



An assassin, currently in node A (A for assassin), is trying to reach her target in node T (T for target). Each arc has one patrolling guard, which she needs to evade. The probability of non-detection by a guard along each arc is indicated on each arc. Detection will immediately trigger an alarm and fail the mission. The assassin wants to maximize her chance of reaching her target silently, i.e. without detection.

8.1 Formulate the problem as a stochastic dynamic programming problem.

8.2 Use stochastic dynamic programming to find all strategies that maximize the assassin's chance of reaching her target silently. What is the probability that the assassin reaches her target silently, if she uses such a strategy?