Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH3231-WE01

Title:
Solitons III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.

Revision:	

ED01/2022

SECTION A

Q1 Compute the dispersion relation for the equation

$$u_t + u_{xxx} + au_{xxxx} + u_{xt} = 0 \ ,$$

where a is a real constant. For which values of a is there physical dissipation? In the case where there is no dissipation, physical or unphysical, compute the phase velocity and group velocity.

Q2 **2.1** If $f = f(x, t)$ is a function, define and calculate the Hirota derivative $D_xD_t(f, f)$.

2.2 If f satisfies $D_xD_t(f, f) = 2$, find an equation that is satisfied by the function $u = \log f$.

2.3 For what values of the constants a, b and c does $f(x, t) = \cosh(ax + bt + c)$ satisfy $D_xD_t(f, f) = 2$?

Q3 **3.1** A functional $F[u]$ is given by

$$F[u] = \int_{-\infty}^{+\infty} f(u, u_x)\, dx \ .$$

Define the functional derivative $\delta F/\delta u$, and derive an expression for this quantity in terms of $\partial f/\partial u$ and $\partial f/\partial u_x$. The boundary conditions on $u(x)$ are that u and u_x tend to zero as $x \to \pm\infty$.

3.2 Compute $\delta F/\delta u$ in the cases

$$\text{(i) } f(u, u_x) = u^2 \ ; \qquad \text{(ii) } f(u, u_x) = u^2 u_x \ .$$

3.3 Your answer to **3.2**(ii) might seem surprisingly simple. Explain this fact by considering the value taken by $F[u]$ in this case.

Q4 **4.1** Use the cyclic property of the trace to show that if a system is described by a matrix Lax pair equation

$$L_t = [M, L]$$

then the quantity $\mathrm{Tr}(L^2)$ is conserved, where Tr denotes the trace.

4.2 If L and M are given in terms of two functions $p(t)$ and $q(t)$ as

$$L = \begin{pmatrix} p & q \\ q & -p \end{pmatrix} \ , \qquad M = \begin{pmatrix} 0 & -q \\ q & 0 \end{pmatrix} \ ,$$

find the equations of motion for p and q that follow from the Lax pair equation. Calculate $\mathrm{Tr}(L^2)$, and use your equations of motion to show explicitly that it is conserved in this case.

SECTION B

Q5 The field $u = u(x,t)$ has energy

$$E = \int_{-\infty}^{+\infty} dx \ \left[\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + 2u^2(1-u^2)^2 \right] .$$

5.1 Use the Bogomol'nyi argument to find a lower bound for the energy in terms of the boundary values of the field u as $x \to \pm\infty$.

5.2 Now assume that $u \to 1$ as $x \to -\infty$ and $u \to 0$ as $x \to +\infty$. Find a numerical value for the lower bound of the energy. Write down the conditions that u must satisfy to saturate this bound, and find the function u that saturates the bound.

Q6 A field u satisfies the sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0$$

and has kinetic energy T and potential energy V given by

$$T = \int_{-\infty}^{+\infty} dx \ \frac{1}{2}u_t^2$$
$$V = \int_{-\infty}^{+\infty} dx \ \left(\frac{1}{2}u_x^2 + 1 - \cos u \right) .$$

6.1 Which boundary conditions should u obey to ensure that the total energy $E = T + V$ be finite?

6.2 Show that with these boundary conditions the total energy is conserved.

6.3 It is given that

$$u(x,t) = 4 \arctan \left[\cot\varphi \cdot \frac{\sin(t \sin\varphi)}{\cosh(x \cos\varphi)} \right] ,$$

is a solution of the sine-Gordon equation, where $\varphi \in (0, 2\pi)$ is a constant. Calculate the kinetic energy T and the potential energy V at times $t = n\tau/4$, where $\tau = 2\pi/\sin\varphi$ and $n = 0, 1, 2, 3, 4$. You may use without proof the identities

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2} , \quad \frac{d}{dx} \tanh x = \frac{1}{\cosh^2 x} = \mathrm{sech}^2 x .$$

Q7 **7.1** If M is a differential operator acting on functions which decay as $x \to \pm\infty$, define $M^\dagger$, the adjoint of M, with respect to the inner product

$$(\phi, \chi) = \int_{-\infty}^{\infty} \phi^*(x)\chi(x)\, dx\,.$$

What does it mean for M to be (a) symmetric (self-adjoint) or (b) antisymmetric (skew) with respect to this inner product?

7.2 If N is a second differential operator also acting on functions which decay as $x \to \pm\infty$, show that $(MN)^\dagger = N^\dagger M^\dagger$.

7.3 Let $D = d/dx$ and $u(x)$ be a real function decaying as $x \to \pm\infty$. Classify each of the following differential operators as either symmetric, antisymmetric, or neither, giving reasons in each case:

$$B_1 = D\,, \quad B_2 = u\,, \quad B_3 = uD\,, \quad B_4 = uD + Du\,, \quad L = D^2 + u\,.$$

7.4 If B is another differential operator and $[L, B]$ is multiplicative (and real), with $L = D^2 + u$ as above, show that L commutes with the symmetric part of B. Explain briefly why this fact might be useful in searching for the equations of the KdV hierarchy.

Q8 Consider the Schrödinger equation

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi(x) = k^2\psi(x)$$

where $V(x) = -b\delta(x)$, b is real, and $\delta(x)$ is the Dirac delta function.

8.1 By integrating the equation from $-\epsilon$ to $+\epsilon$ and taking the limit $\epsilon \to 0$, find the matching condition determining the discontinuity in $\psi'(x)$ at $x = 0$. You can assume that $\psi(x)$ itself is everywhere continuous.

8.2 For $k^2 > 0$, solve the equation for $x < 0$ and $x > 0$ and then apply your matching condition, and the continuity of ψ at $x = 0$, to fix the coefficients of the solution for $x < 0$ in terms of those for $x > 0$. Use this to find a scattering solution, normalised so that the coefficient of e^{ikx} for $x < 0$ is equal to 1. What are the corresponding reflection and transmission coefficients?

8.3 The potential $V(x)$ is now replaced by $W(x) = c\theta(x)$, where $c > 0$ is real and $\theta(x)$ is the Heaviside step function, equal to 1 for $x > 0$ and zero for $x \leq 0$. Derive the matching condition at $x = 0$ in this case, and find the general solution of the equation as in part **8.2**. For what possibly c-dependent values of $k^2 > 0$ does this new equation have a solution which tends to zero as $x \to +\infty$?