Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH3251-WE01

Title: Stochastic Processes III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 Suppose that $(Y_i)_{i\in\mathbb{N}}$ is a sequence of independent and identically distributed non-negative integer valued random variables and N is a non-negative integer valued random variable that is independent of $(Y_i)_{i\in\mathbb{N}}$. Define

$$S_N := \sum_{i=1}^{N} Y_i$$

and let G_N, G_{Y_1}, G_{S_N} be the generating functions of N, Y_1, S_N respectively; that is:

$$G_N(s) = \mathsf{E}(s^N) \quad ; \quad G_{Y_1}(s) = \mathsf{E}(s^{Y_1}) \quad ; \quad G_{S_N}(s) = \mathsf{E}(s^{S_N})$$

for all $s \in (0,1)$.

(a) Prove that $G_{S_N}(s) = G_N(G_{Y_1}(s))$ for all $s \in (0,1)$.

(b) Suppose that a fair coin (equally likely to land H or T) is tossed infinitely many times, and assume that the results of distinct tosses are independent. Let $(Y_i)_{i\in\mathbb{N}}$ be defined by setting $Y_i = 1$ if the ith toss is a T and 0 otherwise. Let $N := \min\{i \geq 1 : Y_i = 1\}$ be the first toss landing T and $S_N := \sum_{i=1}^{N} Y_i$. Calculate

$$G_{S_N}(s) = \mathsf{E}(s^{S_N})$$

for $s \in (0,1)$.

(c) Calculate $G_N(G_{Y_1}(s))$ for $s \in (0,1)$, where N, Y_1 are distributed as in part (b). Do your answers for (b) and (c) contradict part (a)?

Q2 Let X_t be a continuous time Markov process on the state space $\mathcal{I} = \{1,2,3\}$ with Q-matrix

$$Q = \begin{pmatrix} -8 & 4 & 4 \\ 2 & -6 & 4 \\ 2 & 0 & -2 \end{pmatrix}$$

(a) Find the characteristic polynomial of Q and identify the eigenvalues.

(b) Compute $p_{2,3}(t)$ and evaluate $\lim_{t\to\infty} p_{2,3}(t)$.

(c) Find the invariant distribution π of the process.

Q3 Suppose that $(Z_n)_{n\geq 0}$ is a time-homogeneous branching process with $Z_0 \equiv 1$ and offspring distribution

$$p_k = \mathsf{P}(Z_1 = k) = p^k(1-p) \quad ; \quad k \geq 0$$

for some $p \in (0,1)$. Let $\rho := \mathsf{P}(\cup_{n\geq 0}\{Z_n = 0\})$ be the extinction probability.

(a) Calculate ρ (as a function of p) for $p \in (0,1)$.

(b) Show that $\mathsf{P}(Z_n > 0) \leq (\frac{p}{1-p})^n$ for all $n \geq 0$ and all $p \in (0,1)$.

In your answers you should clearly state and carefully apply any result(s) that you use.

Q4 Let N_t be a Poisson process with rate λ. Suppose $0 \leq s < t$ and $0 \leq m \leq n$.

(a) Compute the probability $\mathsf{P}(N_t = n \mid N_s = m)$.

(b) Compute the probability $\mathsf{P}(N_s = m \mid N_t = n)$.

SECTION B

Q5 A standard fair die is tossed repeatedly. Let T be the number of tosses until the sequence $3 - 2 - 1 - 3$ is observed for the first time. Use the appropriate optional stopping theorem to find the expectation $\mathsf{E}(T)$. In your answer you should clearly state and carefully apply any result you use.

Q6 Let $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ be independent simple symmetric random walks starting from $X_0 \equiv 1$ and $Y_0 \equiv -1$ respectively. That is,

$$X_n := 1 + \sum_{i=1}^{n} Z_i \quad ; \quad Y_n := -1 + \sum_{i=1}^{n} W_i$$

where $Z_1, W_1, Z_2, W_2 \dots$ are all independent, each taking values $+1$ with probability $\frac{1}{2}$ and -1 with probability $\frac{1}{2}$.

6.1 Let $T_0 := \min\{n \geq 0 : X_n = 0\}$ be the hitting time of 0 by $(X_n)_{n\geq 0}$. Using generating functions or otherwise, show that $\mathsf{P}(T_0 < \infty) = 1$.

6.2 Using coupling or otherwise, show that Y_n is stochastically dominated by X_n for all $n \geq 0$.

6.3 Define $(\tilde{X}_n, \tilde{Y}_n)_{n\geq 0}$ by letting $(\tilde{X}_n)_{n\geq 0}$ have the same distribution as $(X_n)_{n\geq 0}$ and given $(\tilde{X}_n)_{n\geq 0}$ and $\tilde{T}_0 = \min\{n \geq 0 : \tilde{X}_n = 0\}$, setting

$$\tilde{Y}_n = \begin{cases} -\tilde{X}_n & \text{for } n \leq \tilde{T}_0 \\ \tilde{X}_n & \text{for } n > \tilde{T}_0 \end{cases}$$

Show that this provides a coupling of (X_n, Y_n) for each n. Hence or otherwise show that the total variation distance $d_{TV}(X_n, Y_n)$ converges to 0 as $n \to \infty$.

In your answers you should clearly state and carefully apply any result(s) that you use.

Q7 Let X_t be a continuous time Markov process on a finite state space $\mathcal{I}$. Let Q be its Q-matrix and P_t be its transition matrices. Suppose $f : \mathcal{I} \to \mathbb{R}$ is a function. The matrices Q and P_t act on f as a column vector, namely the functions Qf and $P_t f$ are defined by

$$Qf(x) = \sum_{y \in \mathcal{I}} Q_{x,y} f(y), \quad P_t f(x) = \sum_{y \in \mathcal{I}} p_{x,y}(t) f(y) \quad \text{for } x \in \mathcal{I}.$$

7.1 For $t > 0$ and $x \in \mathcal{I}$, define $u_t(x) = \mathsf{E}[f(X_t) \mid X_0 = x]$. Show that

$$\frac{d}{dt} u_t(x) = Q u_t(x) = \sum_{y \in \mathcal{I}} Q_{x,y} u_t(y).$$

7.2 The function f as above is called harmonic if $P_t f = f$ for every $t \geq 0$. Show that f is harmonic if and only if $Qf = 0$.

7.3 Suppose f is a harmonic function and define $M_t = f(X_t)$. Let $\mathcal{F}_t = \sigma(X_s; s \leq t)$ be the sigma-algebra generated by the process up to time t. Show that M_t is a martingale with respect to filtration $\mathcal{F}_t$, that is,

$$\mathsf{E}[M_t \mid \mathcal{F}_s] = M_s \quad \text{for } s < t.$$

Q8 Let B_t be standard Brownian motion.

8.1 Let $m_t = \min\{B_s; s \leq t\}$ be the minimum of Brownian motion from time 0 to time t. Find a simple expression for

$$\mathsf{P}(m_t \leq a)$$

that only involves the distribution of B_t at time t.

8.2 The zero set of Brownian motion is the set $Z = \{t \geq 0 : B_t = 0\}$. The measure of the zero set is

$$|Z| = \int_0^\infty \mathbf{1}_{\{B_t = 0\}} \, dt.$$

Prove that $|Z| = 0$ almost surely.

8.3 A local maximum of Brownian motion is a time t such that for some $\delta > 0$, $B_s \leq B_t$ for every $s \in (t - \delta, t + \delta)$. Prove that, almost surely, Brownian motion has a local maximum in every interval $[a, b]$ of positive length.

Hint: Recall that almost surely, Brownian motion is not monotone on any interval.