EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH3301-WE01

Title: Mathematical Finance III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 (a) State the Cox–Ross–Rubinstein formula for the price at time $T - t$ of a European call option with strike price K and expiry date T. Explain clearly any notation you use in your formula.

(b) Using your formula in part (a), calculate the price at time 0 of a European call option with strike price $K = 40$ and expiry date $T = 5$ on an underlying risky asset with price S_t that evolves according to the 5-period binomial model with $S_0 = 40, u = 1.2, d = 0.8, p_u = 0.8, p_d = 0.2$ and where the interest rate is compounded discretely at rate $r = 0.1$ per time period.

(c) Explain what happens to the Cox–Ross–Rubinstein price in part (b) for values of $K < 13$.

Q2 (a) Give clear definitions of the European call and put options, explain what is meant by the strike price and expiry date of these options, and specify the contract function for each.

(b) State and prove the put-call parity equation for European options with common strike price K and expiry date T, assuming that risk-free interest is compounded continuously at rate r.

Q3 (a) State Lévy's Theorem for the characterisation of a Brownian motion.

(b) Let $X = (X_t, t \geq 0)$ and $Y = (Y_t, t \geq 0)$ be two independent Brownian motions. Show that the process $Z = (Z_t, t \geq 0)$ defined by the Itô integral

$$Z_t := \int_0^t \cos(s) dX_s + \int_0^t \sin(s) dY_s, \qquad t \geq 0$$

is also a Brownian motion.

(c) What is the probability density function of $U := \int_0^1 e^{-t} dZ_t$?

Q4 Under the Black–Scholes model, let $C(K, T, \sigma, S_0, r)$ be the price of a European call option with strike price K and time to expiry T where the underlying stock has volatility σ and current price S_0, and r is the risk-free rate.

(a) State the Black-Scholes formula for $C(K, T, \sigma, S_0, r)$.

(b) Find $\dfrac{\partial C}{\partial \sigma}$ and show that the price of a European call option, as a function of the volatility parameter $\sigma > 0$, is strictly increasing.

(c) Does the price of a European put option increase with $\sigma > 0$ as well?

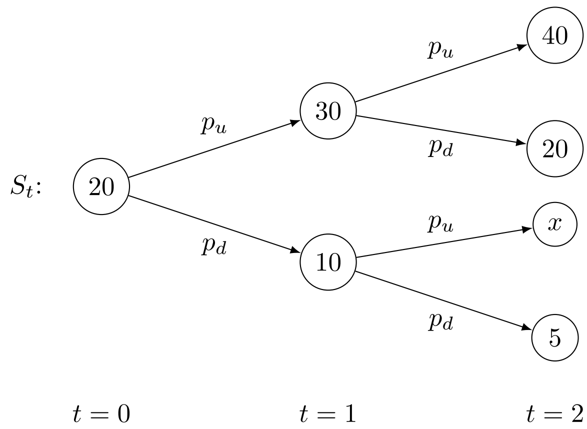
SECTION B

Q5 (a) Define the four main types of barrier option derived from a European-style option with contract function Φ and expiry date T. In each case provide a random variable that represents the random payoff of the barrier option. You may give your answer in terms of the price S_t of the underlying risky asset, and the contract function Φ, but you should define any other notation you use.

(b) Suppose that the price S_t of a risky asset evolves according to the 3-period binomial model, with parameters $S_0 = 216, u = 7/6, d = 5/6, p_u = 1/4, p_d = 3/4$ and that interest is compounded at rate $r = 1/12$ per time period.

Calculate the prices at all times $t = 0, 1, 2, 3$ of a down-and-out barrier (European) put option with strike price $K = 250$, barrier 160, and expiry date $T = 3$.

Q6 Consider the market (B_t, S_t) where $B_t = (5/4)^t$ for $t = 0, 1, 2$, and S_t evolves randomly according to the diagram below, with $p_u = p_d = 1/2$ at each node.



(a) State the conditions on x that guarantee that the market is arbitrage free and calculate the risk-neutral probabilities at every node in this case.

(b) Give the definition of an American put option with strike price K and expiry time T and describe the algorithm for pricing an American put option on this market. (You do not need to prove the correctness of the algorithm.)

(c) Calculate the price at time 0 of an American put option with strike price $K = 20$ and expiry date $T = 2$, as a function of x.

(d) Now suppose that the strike price of the option is $K = 30$. Prove that it is always optimal to exercise at $t = 0$, whatever the value of x (assuming the market is arbitrage free).

Q7 **7.1** Show that if $W = (W_t, t \geq 0)$ is a Brownian motion and $c > 0$, then the process $(c^{-1/2}W_{ct}, t \geq 0)$ is also a Brownian motion.

7.2 Let us use the notation $(x)_+ = \max(x, 0)$ for the rest of this question. Consider

$$H(r, \sigma, T) := \mathbb{E}\left[\left(10 - \max_{0 \leq t \leq T} S_t\right)_+\right]$$

where the underlying stock price evolves as

$$dS_t = rS_t dt + \sigma S_t dW_t, \qquad S_0 = 1$$

where $r, \sigma > 0$ are some constant parameters. Find two functions

$$\tilde{r}(c, r, \sigma) \qquad \text{and} \qquad \tilde{\sigma}(c, r, \sigma)$$

such that for any $c > 0$

$$H(r, \sigma, cT) = H(\tilde{r}, \tilde{\sigma}, T).$$

7.3 Show that if

$$\alpha := \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right) > 0,$$

then

$$\mathbb{E}\left[\left(10 - \max_{0 \leq t \leq T} S_t^{1/2}\right)_+\right] \leq 10^{\alpha/\sigma} e^{-\frac{3}{8}\alpha^2 T} H(r, \sigma, T/4).$$

Q8 Consider a portfolio $(P_t, t \geq 0)$ described by the stochastic differential equation

$$\frac{dP_t}{P_t} = a_t \frac{dS_t^{(1)}}{S_t^{(1)}} + (1 - a_t) \frac{dS_t^{(2)}}{S_t^{(2)}}, \qquad P_0 = 1$$

where $(S^{(1)}, S^{(2)})$ are two assets that evolve according to

$$\frac{dS_t^{(i)}}{S_t^{(i)}} = \mu_i dt + \sigma_i dW_t^{(i)}, \qquad S_0^{(i)} = 1 \qquad i = 1, 2$$

with $(W_t^{(1)})_{t \geq 0}$ and $(W_t^{(2)})_{t \geq 0}$ being two independent Brownian motions. Here $\mu_1, \mu_2, \sigma_1, \sigma_2 > 0$ are some constant parameters, and $(a_t, t \geq 0)$ is a continuous process adapted to $\mathcal{F}_t := \sigma((W_u^{(1)}, W_u^{(2)}), u \in [0, t])$ with the property that $a_t \in [0, 1]$ for all $t \geq 0$.

8.1 Let $R_t = \log P_t$. Express $\mathbb{E}[R_t]$ in terms of $(a_s, s \geq 0)$ and $(\mu_1, \mu_2, \sigma_1, \sigma_2)$.

8.2 Show that there exists some constant $C > 0$ independent of t and $(a_s, s \geq 0)$ such that

$$\mathbb{E}[[R]_t] \leq Ct \qquad \forall t \geq 0.$$

8.3 Fix some $\alpha > 0$ and consider

$$a_t := \frac{(S_t^{(1)})^\alpha}{(S_t^{(1)})^\alpha + (S_t^{(2)})^\alpha}.$$

By finding the distribution of $S_t^{(1)}/S_t^{(2)}$ or otherwise, show that for any $\epsilon \in (0, 1)$,

$$\lim_{t \to \infty} \mathbb{P}(a_t \in [\epsilon, 1 - \epsilon]) = 0.$$

8.4 Let $m := \max_{i=1,2}\{\mu_i - \frac{1}{2}\sigma_i^2\}$. Using a_t defined in the previous part, show that

$$\lim_{t \to \infty} t^{-1} \mathbb{E}[R_t] = m \qquad \text{and} \qquad \lim_{t \to \infty} t^{-2} \mathrm{Var}(R_t) = 0$$

and conclude that R_t/t converges in probability to m as $t \to \infty$.