EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH3391-WE01

Title: Quantum Computing III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.

Revision:	

ED01/2022

SECTION A

Q1 The Toffoli gate, $\hat{T}$, is a 3-qubit gate that acts as a Controlled-Controlled-NOT gate (CCNOT). If the control qubit $|a\rangle$, with $a \in \{0,1\}$, and the control qubit $|b\rangle$,with $b \in \{0,1\}$, are both 1 then the third qubit $|c\rangle$ is negated, $|\bar{c}\rangle = |1-c\rangle$, with $c \in \{0,1\}$, otherwise the third qubit is left untouched, i.e.

$$\hat{T}\,|a,b,c\rangle = \begin{cases} |a,b,1-c\rangle\ , & \text{if } a=b=1\,, \\ |a,b,c\rangle\ , & \text{otherwise.} \end{cases}$$

Write down the $2^3 \times 2^3$ matrix representing the Toffoli gate $\hat{T}$ using the standard basis and show that this operator is unitary.

Q2 Given the matrix

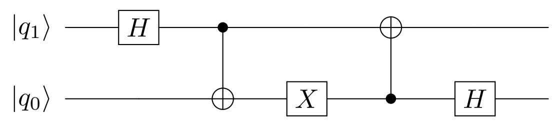
$$\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} - \frac{i}{4} \\ \frac{1}{4} + \frac{i}{4} & \frac{1}{2} \end{pmatrix}\ ,$$

explain why ρ can describe the state of a qubit and deduce whether such state is pure or mixed. Finally compute the von Neumann entropy $S(\rho)$ for this state.

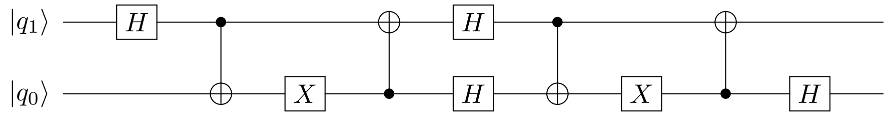
Q3 **3.1** In classical computation we consider functions with n-bit inputs and m-bit outputs. In quantum computing we consider unitary transformations acting on an n-qubit Hilbert space. In this context, what is meant by a *universal gate set* in each case?

3.2 In classical computing {AND, OR, NOT, CNOT} is one example of a universal gate set. Show that {NAND, CNOT} is also a universal gate set.

Q4 **4.1** Calculate the action of the following quantum circuit on computational basis states.



4.2 Use your results from the previous part to give the action of the following quantum circuit on computational basis states.



and draw a simpler circuit with the same action. Your circuit must use only gates from the gate set {H, X, Y, Z, CNOT}.

SECTION B

Q5 Alice and Bob share the two-qubit state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big(|+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle \Big) .$$

5.1 Is the state $|\Psi\rangle$ separable or entangled? Justify your answer.

5.2 What are the possible outcomes, their corresponding probabilities and the final states if Alice measures the observable σ_3 on the first qubit?

5.3 Compute the Schmidt number for the state $|\Psi\rangle$ and the entanglement entropy $S(A)$. How are these two results related to each others? [Hint: Remember that the entanglement entropy is the von Neumann entropy of the reduced density matrix]

Q6 Consider a bipartite system where Alice has two qubits which we will label as system AB, and Charlie has one qubit which we label as system C.

6.1 If the system is in a separable pure state, write an expression for the state and hence give an expression for the density operator $\hat{\rho}$. Calculate the reduced density operator of system AB, $\hat{\rho}_{AB} \equiv \mathrm{Tr}_C(\hat{\rho})$ and evaluate $\mathrm{Tr}(\hat{\rho}_{AB})$ and $\mathrm{Tr}(\hat{\rho}^2_{AB})$.

6.2 For each of the following states

$$\begin{aligned} |\Psi\rangle &= \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle) \\ |\Phi\rangle &= \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle) \end{aligned}$$

calculate the reduced density operator in system AB, and give an interpretation as an ensemble of orthonormal pure states.

Give an example of an observable $\hat{M}$ Charlie could measure which would result in these states in system AB with the same probabilities as in the ensembles you found.

6.3 Suppose Charlie measures the observable $\hat{N} = |0\rangle\langle 1| + |1\rangle\langle 0|$. For each of the initial states $|\Psi\rangle$ and $|\Phi\rangle$, what are the possible final states in system AB, and the probabilities for each outcome?

6.4 If Alice sends her second qubit to Bob, we can consider the system AB as a bipartite system where Alice has one qubit and Bob the other.

For each of the states $|\Psi\rangle$ and $|\Phi\rangle$, explain whether or not it is possible for Charlie to make a local measurement so that with probability 1 the resulting state in system AB:

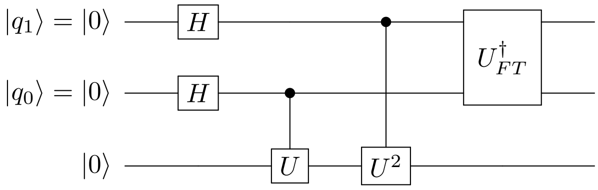
- is entangled.
- is separable.

Q7 The Quantum Fourier Transform is defined as the unitary operator U_{FT} whose action on the computational basis states of an n-qubit Hilbert space is

$$U_{FT}\,|x\rangle = \frac{1}{2^{n/2}} \sum_{y=0}^{2^n-1} e^{2\pi i x y/2^n}\,|y\rangle \ .$$

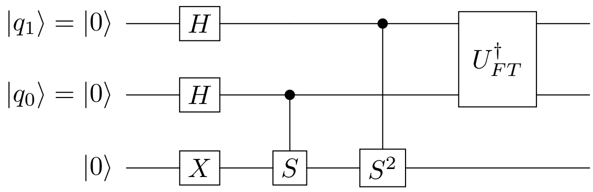
7.1 Evaluate the action of U_{FT} on the computational basis states for the case $n = 2$.

7.2 Suppose we have a quantum circuit



for some unitary $U = V^N$ for some integer N where $V = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$. What can we learn about U by measuring the output values of q_1, q_0 in the computational basis?

7.3 Suppose instead we have the circuit



where $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$. What probabilities would we have for the measured output values (q_1, q_0)?

Q8 **8.1** Consider the operators in a 5 qubit Hilbert space

$$M_0 = X_4 Z_3 Z_2 X_1, \quad M_1 = Z_4 Z_3 X_2 X_0, \quad M_2 = Z_4 X_3 X_1 Z_0,$$

$$M_3 = X_4 X_2 Z_1 Z_0, \quad M_4 = X_3 Z_2 Z_1 X_0.$$

Explain why any simultaneous eigenstate of M_0, M_1, M_2, M_3 is also an eigenstate of M_4 and show that

$$|\bar{0}\rangle = \frac{1}{4}(I + M_0)(I - M_1)(I + M_2)(I - M_3)|00000\rangle,$$

$$|\bar{1}\rangle = \frac{1}{4}(I + M_0)(I - M_1)(I + M_2)(I - M_3)|11111\rangle$$

are both eigenstates of all the M_i with the same eigenvalues.

8.2 Using $|\bar{0}\rangle$ and $|\bar{1}\rangle$ as logical qubit basis states for the code subspace, show that the error subspaces obtained after the single-qubit errors X_2, Y_2 or Z_2 are also eigenspaces of all M_i, and that the code subspace and all three error subspaces are mutually orthogonal.

8.3 Noting that $\{I, X, Y, Z\}$ is a basis for 2×2 matrices, explain how to recover from arbitrary single-qubit errors on qubit 2.

8.4 By symmetry it can be shown (You do not have to show this.) that it is possible to recovery from an arbitrary single-qubit error on any qubit. Is $\bar{Z} = Z_4 Z_3 Z_2 Z_1 Z_0$ a fault tolerant gate implementing the Pauli Z operator on the logical qubit? Justify your answer.