Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH3471-WE01

Title:

Geometry of Mathematical Physics III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 **1.1** State the definition of the group $SU(2)$.

1.2 Check that

$$g(t) := \exp\left(it\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)$$

is an element of $SU(2)$ for all $t \in \mathbb{R}$. Determine for which values of t the group element $g(t)$ is equal to the identity by expanding the exponential.

1.3 Letting $t \in (-1, 1)$, $g(t)$ defines a path in $SU(2)$. Find the associated Lie algebra element γ_1 by computing the tangent vector at $g(0)$. Which Lie algebra element is associated to the path $g(2t)$?

1.4 Find a basis of the Lie algebra of $SU(2)$ and explain your reasoning.

Q2 Let $\boldsymbol{q}$ be a vector which transforms in the fundamental representation of $SU(3)$, and M a matrix which transforms in the adjoint representation of $SU(3)$.

2.1 Give the transformation behaviour of $\boldsymbol{q}$ and M under a group element $g \in SU(3)$.

2.2 For a mechanical system with finitely many degrees of freedom $\boldsymbol{q}$, show that the action

$$S = \int dt\; \dot{\boldsymbol{q}}^\dagger \dot{\boldsymbol{q}} - V(\boldsymbol{q}^\dagger \boldsymbol{q})$$

is invariant and find the conserved quantities.

2.3 Denoting the components of $\boldsymbol{q}$ by q_i and the components of M by M_{ij}, find the action of $SU(3)$ on

i) A vector $\boldsymbol{v}$ with components $v_i = \sum_j M_{ij} q_j$

ii) A matrix Q with components $Q_{ij} = q_i q_j$

iii) A matrix $E = \exp(tM)$ for $t \in \mathbb{R}$

Show that this defines a representation of $SU(3)$ in each of the cases. (You do not need to prove that the action is on a vector space.)

Q3 Let A_μ be a $U(1)$ gauge field, ϕ a scalar field of charge 1, and χ a scalar field of charge -2, with gauge transformations

$$(A_\mu,\ \phi,\ \chi)\ \mapsto\ (A_\mu + \partial_\mu \alpha,\ \ \exp[i\alpha]\phi,\ \ \exp[-2i\alpha]\chi)\ .$$

Show that

$$\chi(z)\exp\left[iq\int_{y,C}^{z} A_\mu(x)dx^\mu\right]\phi(y)^2$$

is gauge invariant for an appropriate value of the integer q that you should find. (The line integral in the exponent is over a curve C from point y to point z.)

Q4 It is given that, in a gauge where $A_0 = 0$, the energy of static field configurations of a gauge field A_μ and a scalar ϕ transforming in the adjoint representation of the gauge group is

$$E = \int d^3x\ \mathrm{tr}\left(\frac{1}{g_{YM}^2}B_iB_i + (D_i\phi)(D_i\phi)\right)\ ,$$

where the spatial indices $i = 1, 2, 3$ are summed over, D_i are covariant derivatives, $B_i = -\frac{1}{2}\epsilon_{ijk}F_{jk}$, and the integral is over space $\mathbb{R}^3$. Find a lower bound for the energy E in terms of an integral over the 2-sphere at spatial infinity. You may use without proof the Bianchi identity $\epsilon_{ijk}D_iF_{jk} = 0$.

SECTION B

Q5 The Lie group $SL(2,\mathbb{C})$ is the group of complex 2×2 matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $\det g = 1$.

5.1 For $g \in SL(2,\mathbb{C})$, show that the action on $\mathbb{R}^4$ with coordinates x_i given by

$$F(g) : \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \mapsto g \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} g^\dagger$$

defines a group homomorphism from $SL(2,\mathbb{C})$ to the Lorentz group. Identify which of the disjoint components of the Lorentz group this homomorphism maps to and explain your reasoning.

5.2 Show that

$$\exp\left(\sum_{i=1}^{3} \alpha_i \sigma_i\right) \in SL(2,\mathbb{C})$$

where $\alpha_i \in \mathbb{C}$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

5.3 The homomorphism between $SL(2,\mathbb{C})$ and the Lorentz group defines a representation of $SL(2,\mathbb{C})$ on $\mathbb{R}^4$. Find the associated Lie algebra representation and describe its action on $\boldsymbol{x}$ (written as a column vector) in terms of the α_i. Verify that this maps the Lie algebra of $SL(2,\mathbb{C})$ to the Lie algebra of the Lorentz group.

Q6 The components of the Faraday tensor in terms of electric and magnetic fields are

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} .$$

6.1 Setting $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, show that $\partial_\mu F_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = 0$. Here $\epsilon^{\mu\nu\rho\sigma}$ is the completely antisymmetric tensor with $\epsilon^{0123} = 1$.

6.2 Write

$$F_{\mu\nu}F_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}$$

in terms of electric and magnetic fields.

6.3 A relativistic point particle is described by its world-line C in space-time with coordinates $\boldsymbol{x} = (x^0, x^1, x^2, x^3)$. Parametrizing C by functions $x^\mu(s)$, the action of a charged particle in the presence of electromagnetic fields is

$$S = \int_C ds \left(\sqrt{-\dot{x}^\mu \dot{x}_\mu} + q\, \dot{x}^\mu A_\mu(\boldsymbol{x}) \right) \tag{1}$$

where $\dot{x}^\mu = \frac{d}{ds}x^\mu$. Show that gauge transformations acting on A_μ are symmetries of this action.

6.4 Find the equations of motion for $x^\mu(s)$ which follow from the action (1) and verify that they are gauge invariant by writing them in terms of electric and magnetic fields.

Q7 The infinitesimal gauge variations of a non-abelian gauge field $A_\mu = A_\mu^\dagger$ and a charged field ϕ are given by

$$\delta_\alpha A_\mu = i[\alpha, A_\mu] + \partial_\mu \alpha \ , \qquad \delta_\alpha \phi = i\alpha\phi \ ,$$

where $\alpha = \alpha^\dagger$ is an infinitesimal element of the Lie algebra of the gauge group, and the gauge transformation with parameter α of any field X is given by

$$X \mapsto X' = X + \delta_\alpha X + O(\alpha^2) \ .$$

7.1 Show that, for any two fields f and g, the infinitesimal gauge variation of their product fg is $\delta_\alpha(fg) = (\delta_\alpha f)g + f(\delta_\alpha g)$.

7.2 Find the infinitesimal gauge variation $\delta_\alpha F_{\mu\nu}$ of the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

and show that

$$\mathcal{L}_{YM} = -\frac{1}{2g_{YM}^2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$$

is invariant under infinitesimal gauge transformations, that is $\delta_\alpha \mathcal{L}_{YM} = 0$.

7.3 Find the infinitesimal gauge variation $\delta_\alpha(D_\mu\phi)$ of the covariant derivative $D_\mu\phi = \partial_\mu\phi - iA_\mu\phi$ of the charged field ϕ. Show that

$$\mathcal{L}_{\text{matter}} = -(D_\mu\phi)^\dagger(D^\mu\phi) - m^2\phi^\dagger\phi$$

is invariant under infinitesimal gauge transformations, that is $\delta_\alpha \mathcal{L}_{\text{matter}} = 0$.

Q8 'Static' (or time-independent) configurations of a charged scalar field ϕ and a gauge field A_j in $n \geq 1$ space dimensions have total energy

$$E[\phi, A] = E_1[\phi, A] + E_2[\phi] + E_3[A] \ ,$$

with

$$\begin{aligned} E_1[\phi, A] &= \int d^n x \ ((\partial_j - iA_j)\phi)^\dagger((\partial_j - iA_j)\phi) \\ E_2[\phi] &= \int d^n x \ V(\phi) \\ E_3[A] &= \int d^n x \ \frac{1}{2g^2}\mathrm{tr}\,(F_{jk}F_{jk}) \ . \end{aligned}$$

Here $j, k = 1, \ldots, n$ are spatial indices, Einstein's summation convention is used, $F_{jk} = \partial_j A_k - \partial_k A_j - i[A_j, A_k]$, and $V(\phi) \geq 0$, with $V(\phi) = 0$ at its minima. We are working in a gauge where $A_0 = 0$.

8.1 Consider a scale transformation

$$\begin{aligned} \phi(x) &\mapsto \phi^\lambda(x) := \phi(\lambda x) \\ A_j(x) &\mapsto A_j^\lambda(x) := \lambda^\beta A_j(\lambda x) \ , \end{aligned}$$

for any $\lambda > 0$. Show that for a suitable choice of the constant β, that you should find, the three contributions to the energy satisfy

$$E_1[\phi^\lambda, A^\lambda] = \lambda^{a_1} E_1[\phi, A] \ , \quad E_2[\phi^\lambda] = \lambda^{a_2} E_2[\phi] \ , \quad E_3[A^\lambda] = \lambda^{a_3} E_3[A]$$

for some constants a_1, a_2 and a_3 that you should also find.

8.2 Using that static solutions of the equations of motion leave the total energy $E[\phi, A]$ stationary under infinitesimal variations of the fields, show that for all static solutions of the equations of motion

$$c_1 E_1[\phi, A] + c_2 E_2[\phi] + c_3 E_3[A] = 0 \ ,$$

for some constants c_1, c_2 and c_3 that you should find.

8.3 Show that if $n = 1$, static solutions of the equations of motion can only exist if the gauge field vanishes, that is $A_j = 0$.

END