Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH4041-WE01

Title:
Partial Differential Equations IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 20%, Section B is worth 60%, and Section C is worth 20%. Within Sections A and B, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 Consider the following Cauchy problem for $u : \mathbb{R}^2 \to \mathbb{R}$

$$\begin{cases} \partial_x u(x,y) + \partial_y u(x,y) = 0, & (x,y) \in \mathbb{R}^2, \\ u(x,-x) = x^2, & x \in \mathbb{R}. \end{cases} \tag{1}$$

(a) Show that the Cauchy curve satisfies the non-characteristic condition.

(b) By differentiating the condition $u(x,-x) = x^2$ with respect to the x-variable, determine $\partial_x u(x,-x)$ and $\partial_y u(x,-x)$.

(c) Solve (1) using the method of characteristics. By direct computation, check that the obtained function satisfies both the PDE and the Cauchy condition.

Q2 Let $v : \mathbb{R}^n \to \mathbb{R}$ be given. Consider the following Cauchy problem associated to the heat equation

$$\begin{cases} \partial_t u(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \mathbb{R}^n \times (0,+\infty), \\ u(\boldsymbol{x},0) = v(\boldsymbol{x}), & \boldsymbol{x} \in \mathbb{R}^n. \end{cases} \tag{2}$$

(a) Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Show that if u is a solution to (2) with initial datum v, then $u_\lambda(\boldsymbol{x},t) := u(\lambda \boldsymbol{x}, \lambda^2 t)$ is a solution to the same PDE, with initial datum $v_\lambda(\boldsymbol{x}) = v(\lambda \boldsymbol{x})$.

(b) Suppose that v is integrable, bounded, continuous and nonnegative. Show that $\lim_{\lambda \to +\infty} u_\lambda(\boldsymbol{x},t) = 0$ for all $(\boldsymbol{x},t) \in \mathbb{R}^n \times (0,+\infty)$. [Hint: use the representation formula via the fundamental solution].

(c) Let v be twice continuously differentiable with compact support. Show that the solution u to (2) is stationary (i.e. $u(\boldsymbol{x},t_1) = u(\boldsymbol{x},t_2)$ for any $t_1, t_2 > 0$ and $\boldsymbol{x} \in \mathbb{R}^n$) if and only if v is harmonic.

(d) Show that the situation in (c) can take place only if v and u are constant zero functions.

SECTION B

Q3 We are aiming to construct weak entropy solutions to the following conservation law.

$$\begin{cases} \partial_t u + \partial_x \left(\frac{1}{4}u^4 + \frac{1}{2}u^2\right) = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where $u_0 : \mathbb{R} \to \mathbb{R}$ is given by

$$u_0(x) = \begin{cases} 1, & x \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch the characteristics associated to this problem. Discuss the need of rarefaction waves and/or shock curves in order to construct an entropy solution.

(b) Using the Rankine-Hugoniot condition, introduce a shock curve that emerges from $(x,t) = (1,0)$. Show that this curve satisfies Lax's entropy condition.

(c) Write down the conditions satisfied by rarefaction waves emerging from the origin. Show the existence of such waves (it is not required to find these waves explicitly).

(d) In order to find a globally defined entropy solutions, notice that a new shock curve needs to be introduced. Determine the ODE and initial condition that must be satisfied by this shock (this might depend on the rarefaction waves; it is not required to find this shock explicitly) [Hint: think about the inverse function theorem].

(e) Find a candidate for the entropy solution to this problem (we allow it to be unbounded) that depends on the rarefaction waves and shocks (it is not required to verify the jump conditions).

Q4 We aim to solve a system of first order PDEs using the method of characteristics. Consider the following Cauchy problem

$$\begin{cases} \partial_x u_1(x,y) - \partial_y u_1(x,y) = u_2(x,y), & (x,y) \in \mathbb{R}^2, \\ \partial_x u_2(x,y) - \partial_y u_2(x,y) = u_1(x,y), & (x,y) \in \mathbb{R}^2, \\ u_1(x,0) = x^2, \ u_2(x,0) = -x^2, & x \in \mathbb{R}, \end{cases} \tag{3}$$

where $u_1, u_2 : \mathbb{R}^2 \to \mathbb{R}$ are the unknown functions. We remark that since the driving vector field is the same in both equations of (3), both equations are governed by the same characteristics.

(a) Determine the Cauchy curve and the Cauchy data associated to this problem.

(b) Write down the ODE system for these characteristics (denoted by $(x(\tau,s), y(\tau,s))$) together with the initial conditions, then solve this system.

(c) Write down the ODE system for the solutions along the flow (denoted by $z_i(\tau,s) := u_i((x(\tau,s), y(\tau,s)), i = 1,2)$, then solve this system.

(d) By inverting the flow, find the solutions $(u_1(x,y), u_2(x,y))$ to (3). By direct computation verify that the functions are indeed solutions.

Q5 Consider the problem

$$\begin{cases} \partial_t u(x,t) + \partial^2_{xx} u(x,t) = 0, & (x,t) \in (0,2\pi) \times (0,+\infty), \\ u(0,t) = u(2\pi,t), & t \geq 0, \\ u(x,0) = u_0(x), & x \in (0,2\pi), \end{cases} \tag{4}$$

where $u_0 : [0,2\pi] \to \mathbb{C}$ is a given smooth function.

(a) Using Fourier series find a candidate for the solution u to (4) in terms of the Fourier coefficients of u_0.

(b) Let $u_0(x) := \tilde{u}_0(x) + \frac{1}{n}\exp(inx)$ for $n \in \mathbb{N}$ and $\tilde{u}_0 : [0,2\pi] \to \mathbb{C}$ is another given smooth function. Find the candidate for the solution to (4) for this initial datum in terms of Fourier coefficients of $\tilde{u}_0$.

(c) Conclude that the problem (4) is ill-posed. [Hint: think about the stability, i.e. continuous dependence of the solution on the data. Use the uniform convergence.]

Q6 We consider the following minimisation problem

$$\min\left\{ \int_0^1 e^t [u'(t)^2 + 2u(t)^2 + 4u(t)]dt : \quad u \in C^1([0,1]),\ u(0) = 0 \right\}.$$

We assume that this problem has a minimiser.

(a) Explain why can we find this minimiser by taking the first variation of the functional?

(b) Take an admissible class of perturbations and derive the ODE and the boundary conditions satisfied by the minimiser.

(c) Solve the problem obtained in (b). Check whether this indeed satisfies the boundary condition.

(d) Compute the minimal value of the original minimisation problem (it can be left in an implicit form as an integral).

SECTION C

Q7 For $n \in \mathbb{N}$, consider $u_n : \mathbb{R} \to (-\pi/2, \pi/2)$, defined as $u_n(x) = \arctan(nx)$. Here arctan denotes for the inverse function of the tan function, i.e. $\tan(\arctan(x)) = x$ for any $x \in \mathbb{R}$ and $\arctan(\tan(\theta)) = \theta$ for any $\theta \in (-\pi/2, \pi/2)$.

(a) Compute the pointwise limit of u_n, as $n \to +\infty$.

(b) Show that u_n converges to its pointwise limit in the sense of distributions, as $n \to +\infty$.

(c) Show that u_n' (the distributional derivative of u_n) converges to $\pi\delta_0$ in the sense of distributions, as $n \to +\infty$. Here δ_0 stands for the Dirac delta mass concentrated at the origin [Hint: $\arctan'(x) = (1+x^2)^{-1}$].

(d) Let $v_n(x) := u_n(x^2)$. Compute the distributional limits of v_n and v_n' (the distributional derivative of v_n), as $n \to +\infty$.