Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH4051-WE01

Title:

General Relativity IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 Consider the spacetime with metric $ds^2 = -dt^2 + 2x dt dx + dy^2 + y^2 d\theta^2$.

(a) Compute the determinant of the metric, and the inverse metric $g^{\mu\nu}$.

(b) Compute the tangent vector to the curve $t = \lambda^2, x = \lambda, y = 1, \theta = \lambda^3$, and find an affine parameter for the curve.

Q2 Suppose that ∇ is a torsion-free connection such that $\nabla_\lambda g_{\mu\nu} = 0$. Derive an expression for the connection coefficients $\Gamma^\lambda_{\mu\nu}$. Show that this connection also satisfies $\nabla_\lambda g^{\mu\nu} = 0$.

Q3 Consider the spacetime with metric

$$ds^2 = -\left(1 - \frac{R}{r}\right) dt^2 + \left(1 - \frac{R}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,.$$

where R is a constant.

(a) Write down two Killing vectors of this spacetime.

(b) An observer at fixed $r = 2R$, $\theta = \phi = 0$, receives two light-rays, one at $t = 0$ and another at $t = T$. How much proper time elapsed in the observer's clock between receiving these two light-rays?

(c) A spaceship sends a light-ray radially from $r = 3R$ towards $r = R$. Compute the amount of coordinate time t it takes the light-ray to travel from $r = 3R$ to $r = 2R$.

Q4 (a) Consider the action for a scalar field $\phi(x)$

$$S[\phi] = \int d^4x \sqrt{-g}\left(\partial_\mu\phi(x)\partial^\mu\phi(x) + R\,V(\phi(x))\right)\,,$$

where R is the Ricci scalar, and $V(\phi(x))$ is some function of $\phi(x)$. What are the equations of motion for ϕ?

(b) The conservation equation reads

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0\,,$$

where dots are derivatives with respect to time as is customary. Determine how the density of a new type of matter satisfying $p = \frac{1}{2}\rho$ scales with $a(t)$. When does this type of matter dominate over normal matter ($p = 0$)?

SECTION B

Q5 (a) For a covector field ω_μ, determine the transformation of $\partial_\mu \omega_\nu$ under changes of coordinates.

(b) Show that $\partial_\mu \omega_\nu - \partial_\nu \omega_\mu$ transform like the components of a $(0,2)$ tensor under changes of coordinates. Determine the relation between this tensor and the covariant derivative $\nabla_\mu \omega_\nu$.

(c) Determine the transformation of $\partial_\lambda(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu)$ under changes of coordinates. Can we construct a non-zero tensor by appropriately symmetrizing or anti-symmetrizing the indices in this object?

(d) By considering $\nabla_{[\mu} \nabla_\nu \omega_{\lambda]}$, show that $R_{\sigma[\lambda\mu\nu]} = 0$. (Recall that square brackets denote antisymmetrization.)

Q6 Consider a spacetime with metric $ds^2 = \frac{1}{z^2}(-dt^2 + dx^2 + dz^2)$, with $z > 0$.

(a) Calculate the Christoffel symbols for this metric.

(b) Show that the semi-circles $t =$ constant, $x^2 + z^2 = R^2$, where R is an arbitrary constant, are geodesics of this metric.

(c) Find the null geodesics for this metric.

Q7 Consider the spacetime with metric

$$ds^2 = -\left(1 - \frac{R}{r}\right) dt^2 + \left(1 - \frac{R}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) .$$

where R is a constant.

(a) Using conserved quantities write an equation for space-like geodesics in the $\theta = \pi/2$ plane. Write your answer as an equation

$$\dot{r}^2 + V(r) = 0$$

where $\dot{r}$ is a derivative with respect to proper length, and $V(r)$ is a function of conserved quantities for geodesics in this spacetime that you should determine.

(b) Change coordinates from (t, r, θ, ϕ) to (u, r, θ, ϕ) with u defined as

$$u = t - r - R \log(r/R - 1) .$$

Write the metric in the new coordinates. Is the metric singular at $r = R$?

(c) Draw the lightcones in these coordinates inside, outside and on the horizon. What happens at the horizon?

Q8 Consider the spacetime with the following metric

$$ds^2 = -dt^2 + a(t)\left(\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right).$$

You may need the following Christoffel symbols:

$$\Gamma^r_{tt} = \Gamma^\theta_{tt} = \Gamma^\phi_{tt} = \Gamma^t_{tt} = 0\,.$$

(a) There are two observers in this spacetime, one is located at $r = 0$ and the other at $r = r_1$. The observer at $r = r_1$ sends light-rays radially to the observer at $r = 0$. Compute the redshift of light received by the observer at $r = 0$.

(b) A spaceship moves in this spacetime along a path with $\theta = \phi = 0$ and

$$r(t) = \frac{1}{2} + \alpha t\,,$$

where α is a constant. What are the constraints on the value of α at $r = \frac{1}{2}$?

(c) Imagine the universe is filled with dust, that is the stress tensor is given by

$$T^{\mu\nu} = \rho u^\mu u^\nu\,,$$

where ρ is the matter density, and u is the four-velocity of matter normalized such that $u^\mu u_\mu = -1$, and we have that the stress tensor is conserved: $\nabla_\mu T^{\mu\nu} = 0$. Do fluid elements follow geodesics? Justify your answer with a computation.