Durham University

# EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
| --- | --- | --- |
| May/June | 2022 | MATH4061-WE01 |

**Title:** Advanced Quantum Theory IV

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| Time: | 3 hours | |
| Additional Material provided: | | |
| Materials Permitted: | | |
| Calculators Permitted: | No | Models Permitted: Use of electronic calculators is forbidden. |

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| Instructions to Candidates: | Answer all questions.<br>Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.<br>Students must use the mathematics specific answer book. |

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| **Revision:** | |

ED01/2022

## SECTION A

**Q1** Consider the following theory of a real scalar field $\phi$

$$S = -\int d^4x \, \left[\frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi + \frac{g}{n!}\phi^n\right].$$

Answer the following questions.

(a) Determine the Euler-Lagrange equations for $\phi$.

(b) Consider a generic Lorentz transformation

$$x^\mu \;\to\; (x')^\mu = \Lambda^\mu{}_\nu \, x^\nu.$$

Show that the action $S$ is invariant under Lorentz transformations. You may use that:

$$d^4x' = |\det(\Lambda)|\, d^4x.$$

(c) Consider the *scale transformation*

$$x \to x' = \lambda x,$$

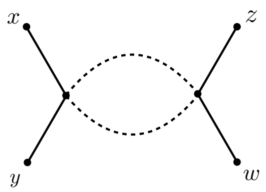
where $\lambda$ is a constant real number. Under this transformation suppose that $\phi$ transforms to the new field $\phi'$ as

$$\phi(x) \to \phi'(x') = \lambda^{-\Delta}\phi(x).$$

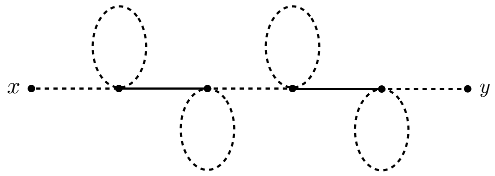
Show that if $g = 0$ the action $S$ is invariant under a scale transformation if $\Delta = 1$.

Taking $\Delta = 1$, for which value of $n$ is the action $S$ invariant under a scale transformation for non-zero $g$?

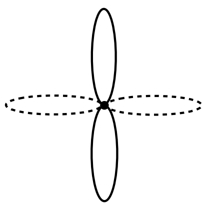
**Q2** Four Feynman diagrams A, B, C and D for a theory of two real scalar fields $\phi_1$ and $\phi_2$ are given below. The solid lines represent the Feynman propagator of the field $\phi_1$ and the dashed lined the Feynman propagator of the field $\phi_2$.



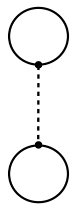

**A**




**B**



**C**



**D**

Answer the following questions.

(a) State which diagrams are vacuum bubble diagrams.

(b) For each diagram, write down the interaction part of the Lagrangian (up to a constant) from which the diagram follows.

(c) Determine the symmetry factor of each diagram.

(d) State the order of each diagram in the coupling constant.

(e) Are any of the diagrams disconnected?

**Q3** Consider a 4-dimensional field theory of two scalar fields $\phi_1$ and $\phi_2$ with the following action

$$S = -\int d^4x \left\{ \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 + \frac{1}{2}m_1^2\phi_1^2 + \frac{1}{2}m_2^2\phi_2^2 + \frac{\lambda}{4!}\left(\phi_1^2 + \phi_2^2\right)^2 \right\} . \quad (1)$$

(a) Write down an expression for the vacuum-vacuum persistent amplitude (the generating functional $Z[J_1, J_2]$) in the presence of two sources $J_1$ and $J_2$ coupled to $\phi_1$ and $\phi_2$, respectively. Define the terms that appear in your expression. What is the condition on $Z[0, 0]$?

(b) Express the two-point functions

$$\langle 0| T\left\{\hat{\phi}_1(x)\hat{\phi}_1(y)\right\}|0\rangle$$
$$\langle 0| T\left\{\hat{\phi}_2(x)\hat{\phi}_2(y)\right\}|0\rangle$$
$$\langle 0| T\left\{\hat{\phi}_1(x)\hat{\phi}_2(y)\right\}|0\rangle$$
$$\langle 0| T\left\{\hat{\phi}_1(x)\hat{\phi}_1(y)\hat{\phi}_1(z)\right\}|0\rangle$$

as functional derivatives of $Z[J_1, J_2]$. Do not take the derivatives at this point.

(c) Recall that the action of a single free field theory is

$$S_0 = -\int d^4x \left\{ \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 \right\}$$

and its generating functional is

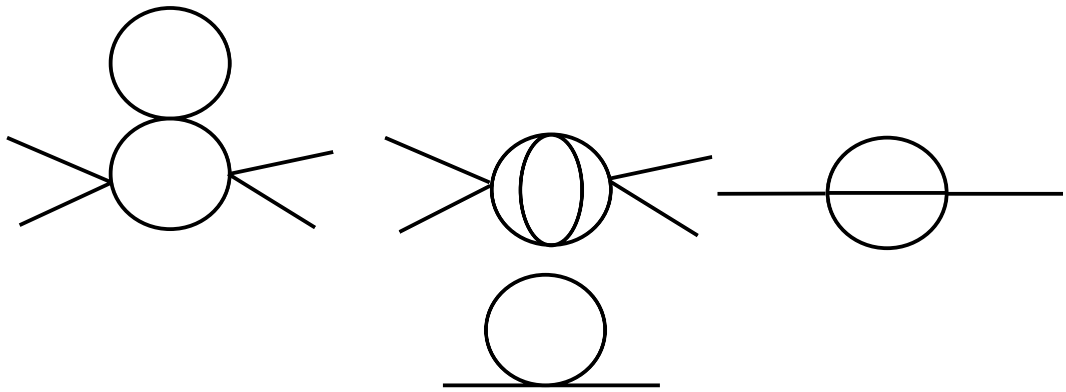
$$Z_0[J] = \mathcal{N} \exp\left[\frac{1}{2}\int d^4x d^4y J(x) G_F(x, y) J(y)\right] ,$$

where $G_F$ is the Feynman propagator (or Green's function) of $\phi$ and $\mathcal{N}$ is a normalization factor. Use this information to write down an expression of the generating functional $Z_0[J_1, J_2]$ of the free field theory for two scalars. Then, express $Z[J_1, J_2]$ of the full action given by Equation (1) as formal functional derivatives acting on $Z_0[J_1, J_2]$. Do not expand your expression and do not take the functional derivatives.

**Q4** (a) Explain the difference between regularization and renormalization. How do you regularize a divergent Feynman diagram?

(b) Explain what it means that a given diagram is superficially divergent.

(c) Find the superficial degree of divergence for each of the following four diagrams of a scalar field theory with $\lambda\phi^4$ interaction in a general spacetime dimension $d$. For $d = 4$, state which diagrams contribute to the normalization of the mass $m$ and which contribute to the normalization of the coupling constant $\lambda$.



## SECTION B

**Q5** Consider a theory of a complex scalar field $\phi$ with Lagrangian density

$$S[\phi, \phi^*] = \int d^4x\, \mathcal{L} = -\int d^4x\, \left[(\partial_\mu \phi^*)\, (\partial^\mu \phi) + V(\phi^*\phi)\right],$$

where $V$ can be any function of $\phi^*\phi$.

(a) Explain why the theory has a conserved current which may be written as

$$j^\mu = i\,(\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi)\,.$$

Determine the corresponding conserved charge $Q$.

(b) In the following take $V(\phi^*\phi) = m^2\, \phi^*\phi$. In this theory, upon canonical quantisation the operator valued field $\hat{\phi}$ can be expressed in the form

$$\hat{\phi}(t, \boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}} \left[\hat{a}_{\boldsymbol{k}} \exp\left(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega_{\boldsymbol{k}} t\right) + \hat{b}^\dagger_{\boldsymbol{k}} \exp\left(-i\boldsymbol{k}\cdot\boldsymbol{x} + i\omega_{\boldsymbol{k}} t\right)\right],$$

where $\omega^2_{\boldsymbol{k}} = |\boldsymbol{k}|^2 + m^2$ and

$$\left[\hat{a}_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{k}'}\right] = (2\pi)^3\, \delta^{(3)}\left(\boldsymbol{k} - \boldsymbol{k}'\right), \qquad \left[\hat{b}_{\boldsymbol{k}}, \hat{b}^\dagger_{\boldsymbol{k}'}\right] = (2\pi)^3\, \delta^{(3)}\left(\boldsymbol{k} - \boldsymbol{k}'\right),$$

where all other commutators are vanishing. Derive the unique expression for the operator $\hat{Q}$ associated to the charge $Q$ in terms of $\hat{a}_{\boldsymbol{k}}$ and $\hat{b}_{\boldsymbol{k}}$ which has finite eigenvalues. Interpret $\hat{Q}$ in terms of charges of the particles in the theory.

**Q6** Consider a theory of three real scalar fields $\phi_1$, $\phi_2$ and $\phi_3$ with action

$$S = -\int d^4x\, \left\{\frac{1}{2}\left(\sum_{i=1}^{3} \partial_\mu \phi_i(x)\, \partial^\mu \phi_i(x) + \sum_{i=1}^{3} m_i^2\, \phi_i(x)^2\right) + \lambda\, \phi_1(x)\, \phi_2(x)\, \phi_3(x)\right\}.$$

(a) Write down the Feynman rules for this theory.

(b) Draw all vacuum bubble diagrams up to and including order $\lambda^2$.

(c) Draw all Feynman diagrams contributing to the time-ordered vacuum expectation value

$$\langle 0|\, T\{\phi_1(x)\, \phi_2(y)\, \phi_3(z)\}\, |0\rangle$$

up to and including order $\lambda^2$. Determine the corresponding analytic expressions.

(d) Evaluate

$$\langle 0|\, T\{\phi_1(x)\, \phi_1(y)\, \phi_1(z)\}\, |0\rangle$$

at all orders in $\lambda$.

**Q7** Consider a 4-dimensional scalar field theory with action

$$S = -\int d^4x \left\{ \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + g\phi^3 \right\} .$$

(a) Write down an expression of the generating functional $Z[J]$ as a functional derivative of the free-field generating functional $Z_0[J]$ to first order in the coupling constant $g$.

(b) Carry out the functional derivatives of the interaction term using the diagrammatic technique (a cross for the external source, etc.) to find an expression of $Z[J]$ including terms that are first order in $g$.

(c) Using the expression you obtained in item (b), find the interacting 3-point function defined as

$$\left[ \frac{\delta^3 Z[J]}{\delta J(x_1) \delta J(x_2) \delta J(x_3)} / Z[J] \right]_{J=0}$$

including terms that are first order in $g$. Write your final answer in terms of the Feynman propagator $G_F$.

**Q8** (a) The string worldsheet energy-momentum tensor is given by

$$T_{\alpha\beta} = \frac{1}{2}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{4}h_{\alpha\beta}h^{\gamma\rho}\partial_\gamma X^\mu \partial_\rho X_\mu = 0\,.$$

Show that the components of the energy-momentum tensor in the flat gauge

$$h_{\alpha\beta} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

are

$$T_{00} = T_{11} = \frac{1}{2}\left(\frac{dX_L}{d\sigma_+}\right)^2 + \frac{1}{2}\left(\frac{dX_R}{d\sigma_-}\right)^2\,,$$
$$T_{01} = T_{10} = \frac{1}{2}\left(\frac{dX_L}{d\sigma_+}\right)^2 - \frac{1}{2}\left(\frac{dX_R}{d\sigma_-}\right)^2\,,$$

where $X^\mu = X^\mu_L(\sigma_+) + X^\mu_R(\sigma_-)$ and $\sigma_\pm = \tau \pm \sigma$. Also we write $Z^2$ for $Z_\mu Z^\mu$.

(b) The generic solution to the equation of motion is given by

$$X^\mu_R = \frac{1}{2}x^\mu + \frac{1}{4\pi T}\sigma_- p^\mu + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in(\tau-\sigma)}\,,$$

and a similar expression for $X^\mu_L$.

(i) Introduce the variable $z = e^{i\sigma_-}$ and rewrite $X^\mu_R$ in terms of $z$. Then, compute $\partial_z X^\mu_R$. Hint: use the definition of $\alpha^\mu_0 = \frac{p^\mu}{\sqrt{4\pi T}}$.

(ii) Use the algebra satisfied by $\alpha^\mu_n$:

$$[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n=0}\eta^{\mu\nu}$$

to show that

$$\langle 0|\partial_z X^\mu_R \partial_w X^\nu_R|0\rangle = -\frac{1}{4\pi T}\frac{\eta^{\mu\nu}}{(z-w)^2}\,,$$

where $|0\rangle$ is the vacuum with vanishing momentum, i.e., $\alpha^\mu_n|0\rangle = 0$ for all $n \geq 0$. Hint: you may need to use the identity $\sum_{n=1}^{\infty} nx^n = \frac{x}{(x-1)^2}$ .