Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH4091-WE01

Title:
Stochastic Processes IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 20%, Section B is worth 60%, and Section C is worth 20%. Within Sections A and B, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 Suppose that $(Y_i)_{i\in\mathbb{N}}$ is a sequence of independent and identically distributed non-negative integer valued random variables and N is a non-negative integer valued random variable that is independent of $(Y_i)_{i\in\mathbb{N}}$. Define

$$S_N := \sum_{i=1}^{N} Y_i$$

and let G_N, G_{Y_1}, G_{S_N} be the generating functions of N, Y_1, S_N respectively; that is:

$$G_N(s) = \mathsf{E}(s^N) \quad ; \quad G_{Y_1}(s) = \mathsf{E}(s^{Y_1}) \quad ; \quad G_{S_N}(s) = \mathsf{E}(s^{S_N})$$

for all $s \in (0,1)$.

(a) Prove that $G_{S_N}(s) = G_N(G_{Y_1}(s))$ for all $s \in (0,1)$.

(b) Suppose that a fair coin (equally likely to land H or T) is tossed infinitely many times, and assume that the results of distinct tosses are independent. Let $(Y_i)_{i\in\mathbb{N}}$ be defined by setting $Y_i = 1$ if the ith toss is a T and 0 otherwise. Let $N := \min\{i \geq 1 : Y_i = 1\}$ be the first toss landing T and $S_N := \sum_{i=1}^{N} Y_i$. Calculate

$$G_{S_N}(s) = \mathsf{E}(s^{S_N})$$

for $s \in (0,1)$.

(c) Calculate $G_N(G_{Y_1}(s))$ for $s \in (0,1)$, where N, Y_1 are distributed as in part (b). Do your answers for (b) and (c) contradict part (a)?

Q2 Let X_t be a continuous time Markov process on the state space $\mathcal{I} = \{1,2,3\}$ with Q-matrix

$$Q = \begin{pmatrix} -8 & 4 & 4 \\ 2 & -6 & 4 \\ 2 & 0 & -2 \end{pmatrix}$$

(a) Find the characteristic polynomial of Q and identify the eigenvalues.

(b) Compute $p_{2,3}(t)$ and evaluate $\lim_{t\to\infty} p_{2,3}(t)$.

(c) Find the invariant distribution π of the process.

SECTION B

Q3 A standard fair die is tossed repeatedly. Let T be the number of tosses until the sequence $3-2-1-3$ is observed for the first time. Use the appropriate optional stopping theorem to find the expectation $\mathsf{E}(T)$. In your answer you should clearly state and carefully apply any result you use.

Q4 Let $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ be independent simple symmetric random walks starting from $X_0 \equiv 1$ and $Y_0 \equiv -1$ respectively. That is,

$$X_n := 1 + \sum_{i=1}^{n} Z_i \quad ; \quad Y_n := -1 + \sum_{i=1}^{n} W_i$$

where $Z_1, W_1, Z_2, W_2 \dots$ are all independent, each taking values $+1$ with probability $\frac{1}{2}$ and -1 with probability $\frac{1}{2}$.

4.1 Let $T_0 := \min\{n \geq 0 : X_n = 0\}$ be the hitting time of 0 by $(X_n)_{n\geq 0}$. Using generating functions or otherwise, show that $\mathsf{P}(T_0 < \infty) = 1$.

4.2 Using coupling or otherwise, show that Y_n is stochastically dominated by X_n for all $n \geq 0$.

4.3 Define $(\tilde{X}_n, \tilde{Y}_n)_{n\geq 0}$ by letting $(\tilde{X}_n)_{n\geq 0}$ have the same distribution as $(X_n)_{n\geq 0}$ and given $(\tilde{X}_n)_{n\geq 0}$ and $\tilde{T}_0 = \min\{n \geq 0 : \tilde{X}_n = 0\}$, setting

$$\tilde{Y}_n = \begin{cases} -\tilde{X}_n & \text{for } n \leq \tilde{T}_0 \\ \tilde{X}_n & \text{for } n > \tilde{T}_0 \end{cases}$$

Show that this provides a coupling of (X_n, Y_n) for each n. Hence or otherwise show that the total variation distance $d_{TV}(X_n, Y_n)$ converges to 0 as $n \to \infty$.

In your answers you should clearly state and carefully apply any result(s) that you use.

Q5 Let X_t be a continuous time Markov process on a finite state space $\mathcal{I}$. Let Q be its Q-matrix and P_t be its transition matrices. Suppose $f : \mathcal{I} \to \mathbb{R}$ is a function. The matrices Q and P_t act on f as a column vector, namely the functions Qf and $P_t f$ are defined by

$$Qf(x) = \sum_{y\in\mathcal{I}} Q_{x,y} f(y), \quad P_t f(x) = \sum_{y\in\mathcal{I}} p_{x,y}(t) f(y) \quad \text{for } x \in \mathcal{I}.$$

5.1 For $t > 0$ and $x \in \mathcal{I}$, define $u_t(x) = \mathsf{E}[f(X_t) \mid X_0 = x]$. Show that

$$\frac{d}{dt} u_t(x) = Q u_t(x) = \sum_{y\in\mathcal{I}} Q_{x,y} u_t(y).$$

5.2 The function f as above is called harmonic if $P_t f = f$ for every $t \geq 0$. Show that f is harmonic if and only if $Qf = 0$.

5.3 Suppose f is a harmonic function and define $M_t = f(X_t)$. Let $\mathcal{F}_t = \sigma(X_s; s \leq t)$ be the sigma-algebra generated by the process up to time t. Show that M_t is a martingale with respect to filtration $\mathcal{F}_t$, that is,

$$\mathsf{E}[M_t \mid \mathcal{F}_s] = M_s \quad \text{for } s < t.$$

Q6 Let B_t be standard Brownian motion.

6.1 Let $m_t = \min\{B_s; s \le t\}$ be the minimum of Brownian motion from time 0 to time t. Find a simple expression for

$$\mathsf{P}(m_t \le a)$$

that only involves the distribution of B_t at time t.

6.2 The zero set of Brownian motion is the set $Z = \{t \ge 0 : B_t = 0\}$. The measure of the zero set is

$$|Z| = \int_0^\infty \mathbf{1}_{\{B_t=0\}}\, dt.$$

Prove that $|Z| = 0$ almost surely.

6.3 A local maximum of Brownian motion is a time t such that for some $\delta > 0$, $B_s \le B_t$ for every $s \in (t-\delta, t+\delta)$. Prove that, almost surely, Brownian motion has a local maximum in every interval $[a, b]$ of positive length.

Hint: Recall that almost surely, Brownian motion is not monotone on any interval.

SECTION C

Q7 **7.1** Suppose that $(Z_n)_{n\ge0}$ is a single-type time-homogeneous branching process with $Z_0 \equiv 1$ and offspring distribution with generating function $\varphi(s) := \mathsf{E}(s^{Z_1})$ (defined for all s such that s^{Z_1} is integrable).

(i) If $\varphi_n(s) := \mathsf{E}(s^{Z_n})$ is the generating function of Z_n, prove the identity

$$\varphi_{n+1}(s) = \varphi(\varphi_n(s)) = \varphi_n(\varphi(s)) \quad \forall s \in (0,1)\ ,\ \forall n \ge 1.$$

(ii) Suppose that Z_1 is integrable and $\mathsf{E}(Z_1) = m$. Give, with proof, an expression for $\mathsf{E}(Z_n)$ for $n \ge 1$ in terms of m and n.

(iii) Suppose that the offspring distribution is Poisson with parameter $\lambda > 0$, that is, $\mathsf{P}(Z_1 = k) = \exp(-\lambda)\frac{\lambda^k}{k!}$ for $k \ge 0$. For which values of λ is the survival probability $\mathsf{P}(\cap_{n\ge0}\{Z_n > 0\})$ strictly positive? In your answer you should clearly state and apply any result that you use.

7.2 Suppose that $(Z_n^1, Z_n^2)_{n\ge0}$ is a two-type time-homogeneous branching process whose offspring distribution has generating function

$$f^1(s_1,s_2) = \mathsf{E}\big(s_1^{Z_1^1} s_2^{Z_1^2} \,|\, (Z_0^1, Z_0^2) = (1,0)\big) = \frac{1}{4}s_1^2 s_2^2 + \frac{1}{2}s_1^2 + \frac{1}{4};$$
$$f^2(s_1,s_2) = \mathsf{E}\big(s_1^{Z_1^1} s_2^{Z_1^2} \,|\, (Z_0^1, Z_0^2) = (0,1)\big) = \frac{1}{2}s_1 s_2 + \frac{1}{2}s_2^2.$$

Calculate the extinction probability

$$\rho^1 = \mathsf{P}\left(\bigcup_n \{(Z_n^1, Z_n^2) = (0,0)\} \,|\, (Z_0^1, Z_0^2) = (1,0)\right),$$

clearly stating and applying any results that you use.

Hint: it may help to first determine $\mathsf{P}(\cup_n\{(Z_n^1, Z_n^2) = (0,0)\} \,|\, (Z_0^1, Z_0^2) = (0,1))$.