Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH41220-WE01

Title: Analysis V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	
	Answer all questions. Section A is worth 20%, Section B is worth 60%, and Section C is worth 20%. Within Sections A and B, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

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SECTION A

Q1 Let $E \subseteq \mathbb{R}$ be measurable.

1.1 State what it means for an extended real-valued function $f : E \to \mathbb{R} \cup \{\infty\}$ to be measurable.

1.2 By using the fact that the collection of measurable sets in $\mathbb{R}$ is an algebra, prove that any finite intersection of measurable sets is measurable.

1.3 For a finite family $\{f_k\}_{k=1}^n$ of measurable functions $f_k : E \to \mathbb{R}$, we define the function $g : E \to \mathbb{R}$ as

$$g(x) = \min\{f_1, f_2, \dots, f_n\}(x) = \min\{f_1(x), f_2(x), \dots, f_n(x)\}.$$

Prove that the function g is measurable.

Q2 Let H be an inner product space with inner product $\langle \cdot, \cdot \rangle$. Let the norm derived from $\langle \cdot, \cdot \rangle$ be denoted by $\| \cdot \|$.

2.1 Prove that for any $x, y \in H$,

$$\|x - y\|^2 = \|x\|^2 - 2\mathrm{Re}(\langle x, y \rangle) + \|y\|^2,$$

where $\mathrm{Re}(z)$ denotes the real part of $z \in \mathbb{C}$.

2.2 State and prove Bessel's Inequality for a Hilbert space H and an orthonormal set $U \subset H$. (You may use the expression given in part **2.1**).

2.3 State the additional assumptions that are required on U in order to achieve equality.

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SECTION B

Q3 **3.1** Let $n \in \mathbb{N}$ and $(f_n)_n$ be a sequence of functions, $f_n : \mathbb{R} \to \mathbb{R}$. State what it means that the sequence $(f_n)_n$ converges uniformly to a function $f : \mathbb{R} \to \mathbb{R}$.

3.2 For $n \in \mathbb{N}$, let

$$g_n(x) = \frac{1}{n^{1/2}} \cdot \chi_{[n,\infty)}(x), \quad x \in \mathbb{R}.$$

(i) Prove that the sequence of functions $(g_n)_n$ converges uniformly to the function

$$g(x) = 0, \quad x \in \mathbb{R}.$$

(ii) Does Fatou's Lemma apply to the sequence of functions $(g_n)_n$? If so, then state the result of Fatou's Lemma for the sequence of functions $(g_n)_n$ given above. If not, then explain why not.

(iii) Does the Monotone Convergence Theorem apply to the sequence of functions $(g_n)_n$? If so, then state the result of the Monotone Convergence Theorem for the sequence of functions $(g_n)_n$ given above. If not, then explain why not.

Q4 Let $E \subseteq \mathbb{R}$ be measurable.

4.1 State what it means for a function $f : E \to \mathbb{R}$ to be integrable.

4.2 State the Lebesgue Dominated Convergence Theorem.

4.3 Prove the following claim: if $f : \mathbb{R} \to \mathbb{R}$ is non-negative and integrable, then

$$\lim_{n\to\infty} \int_{1-\sqrt{n}}^{1+\sqrt{n}} f = \int_{\mathbb{R}} f.$$

4.4 Consider the sequence of functions $(f_k)_k$, $k \in \mathbb{N}$, where $f_k : \mathbb{R} \to \mathbb{R}$ is given by

$$f_k(x) = k \cdot \chi_{[0,1/k]}(x).$$

Does the Lebesgue Dominated Convergence Theorem apply to the sequence $(f_k)_k$? Give a full justification of your response.

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Q5 Let $N \in \mathbb{N}$. For $k \in \{1, \ldots, N\}$, let X_k be normed linear spaces with corresponding norms denoted by $\|\cdot\|_{X_k}$. Consider the space

$$Z = \bigcap_{k=1}^{N} X_k$$

where we assume that all the X_k have the same operations of addition and scalar multiplication so that Z is a linear space.
Define the function $\|\cdot\|_Z : Z \to \mathbb{R}$ as

$$\|w\|_Z = \sum_{k=1}^{N} \|w\|_{X_k}, \quad w \in Z.$$

5.1 Prove that $\|\cdot\|_Z$ defines a norm on Z.

5.2 Give an explicit example of $(Z, \|\cdot\|_Z)$ to show that it is not always a Banach space and justify your response briefly.

5.3 Let $E = [0, \pi]$. Let $X_1 = L^2(E)$ where $\|\cdot\|_{X_1}$ is the usual L^2-norm, $X_2 = L^3(E)$ where $\|\cdot\|_{X_2}$ is the usual L^3-norm, and $X_3 = L^6(E)$ where $\|\cdot\|_{X_3}$ is the usual L^6-norm. For $k \in \mathbb{N}$, consider the functions

$$g_k(x) := \frac{(\sin(\sqrt{k}x))^3}{k^{7/6}} \cdot \chi_{[0,\pi/\sqrt{k}]}.$$

Does the sequence $(g_k)_k$ converge in $Z = L^2(E) \cap L^3(E) \cap L^6(E)$ with respect to $\|\cdot\|_Z$? Give a full justification of your response.

Q6

6.1 Prove that the function

$$\langle f, g \rangle = \int_{\mathbb{R}} f\overline{g}$$

is well defined for $f, g \in L^2(\mathbb{R})$ and gives rise to an inner product on $L^2(\mathbb{R})$.

6.2 Let the norm derived from $\langle \cdot, \cdot \rangle$ be denoted by $\|\cdot\|$. Let $E \subset \mathbb{R}$ be measurable. Let $S \subset L^2(\mathbb{R})$ be the space of functions that vanish almost everywhere in $\mathbb{R} \setminus E$. For $f \in L^2(\mathbb{R})$, prove that

$$\|f - g\| \geq \|f - \chi_E \cdot f\|$$

for all $g \in S$.

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SECTION C

Q7 **7.1** For $k \in \mathbb{N}$, consider the functions $h_k : (1,2) \to \mathbb{R}$ defined by

$$h_k(x) = \frac{1}{k^3(x-1)^2}.$$

(i) Determine the pointwise limit $h(x) = \lim_{k\to\infty} h_k(x)$. Does $(h_k)_{k\in\mathbb{N}}$ converge uniformly to h on $(1,2)$? Justify your response.

(ii) State Egoroff's Theorem.

(iii) Prove that Egoroff's Theorem applies to $(h_k)_{k\in\mathbb{N}}$ and explicitly construct a closed set $F \subset (1,2)$ such that $(h_k)_{k\in\mathbb{N}}$ converges uniformly to h on F and $m(F) \geq \frac{1}{4}$ (where m denotes the Lebesgue measure).

7.2 (i) State Vitali's Theorem.

(ii) To obtain the conclusion of Vitali's Theorem, why is it necessary to assume that the outer measure of the set is non-zero?

(iii) Give an explicit example of a function $f : \mathbb{R} \to \mathbb{R}$ that is not measurable but such that both f^2 and $|f|$ are measurable.

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