

EXAMINATION PAPER

Examination Session: May/June

2022

Year:

Exam Code:

MATH41420-WE01

Title:

Solitons V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 20%, Section B is worth 60%, and Section C is worth 20%. Within Sections A and B, all questions carry equal marks.
	Students must use the mathematics specific answer book.

Revision:

SECTION A

 $\mathbf{Q1}$ Compute the dispersion relation for the equation

$$u_t + u_{xxx} + au_{xxxx} + u_{xt} = 0 ,$$

where a is a real constant. For which values of a is there physical dissipation? In the case where there is no dissipation, physical or unphysical, compute the phase velocity and group velocity.

 ${\bf Q2}~~{\bf 2.1}~$ Use the cyclic property of the trace to show that if a system is described by a matrix Lax pair equation

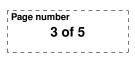
$$L_t = [M, L]$$

then the quantity $Tr(L^2)$ is conserved, where Tr denotes the trace.

2.2 If L and M are given in terms of two functions p(t) and q(t) as

$$L = \begin{pmatrix} p & q \\ q & -p \end{pmatrix}$$
, $M = \begin{pmatrix} 0 & -q \\ q & 0 \end{pmatrix}$,

find the equations of motion for p and q that follow from the Lax pair equation. Calculate $Tr(L^2)$, and use your equations of motion to show explicitly that it is conserved in this case.



SECTION B

Q3 The field u = u(x, t) has energy

$$E = \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + 2u^2(1-u^2)^2 \right] .$$

- **3.1** Use the Bogomol'nyi argument to find a lower bound for the energy in terms of the boundary values of the field u as $x \to \pm \infty$.
- **3.2** Now assume that $u \to 1$ as $x \to -\infty$ and $u \to 0$ as $x \to +\infty$. Find a numerical value for the lower bound of the energy. Write down the conditions that u must satisfy to saturate this bound, and find the function u that saturates the bound.
- $\mathbf{Q4}$ A field u satisfies the sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0$$

and has kinetic energy T and potential energy V given by

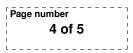
$$T = \int_{-\infty}^{+\infty} dx \, \frac{1}{2} u_t^2$$
$$V = \int_{-\infty}^{+\infty} dx \, \left(\frac{1}{2} u_x^2 + 1 - \cos u\right) \, dx$$

- **4.1** Which boundary conditions should u obey to ensure that the total energy E = T + V be finite?
- 4.2 Show that with these boundary conditions the total energy is conserved.
- 4.3 It is given that

$$u(x,t) = 4 \arctan\left[\cot \varphi \cdot \frac{\sin(t\sin \varphi)}{\cosh(x\cos \varphi)}\right]$$

is a solution of the sine-Gordon equation, where $\varphi \in (0, 2\pi)$ is a constant. Calculate the kinetic energy T and the potential energy V at times $t = n\tau/4$, where $\tau = 2\pi/\sin\varphi$ and n = 0, 1, 2, 3, 4. You may use without proof the identities

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2} , \quad \frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x .$$



Q5 5.1 If M is a differential operator acting on functions which decay as $x \to \pm \infty$, define M^{\dagger} , the adjoint of M, with respect to the inner product

$$(\phi, \chi) = \int_{-\infty}^{\infty} \phi^*(x) \chi(x) \, dx \, .$$

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What does it mean for M to be (a) symmetric (self-adjoint) or (b) antisymmetric (skew) with respect to this inner product?

- **5.2** If N is a second differential operator also acting on functions which decay as $x \to \pm \infty$, show that $(MN)^{\dagger} = N^{\dagger}M^{\dagger}$.
- **5.3** Let D = d/dx and u(x) be a real function decaying as $x \to \pm \infty$. Classify each of the following differential operators as either symmetric, antisymmetric, or neither, giving reasons in each case:

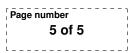
$$B_1 = D$$
, $B_2 = u$, $B_3 = uD$, $B_4 = uD + Du$, $L = D^2 + u$.

- **5.4** If B is another differential operator and [L, B] is multiplicative (and real), with $L = D^2 + u$ as above, show that L commutes with the symmetric part of B. Explain briefly why this fact might be useful in searching for the equations of the KdV hierarchy.
- Q6 Consider the Schrödinger equation

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi(x) = k^2\psi(x)$$

where $V(x) = -b\delta(x)$, b is real, and $\delta(x)$ is the Dirac delta function.

- **6.1** By integrating the equation from $-\epsilon$ to $+\epsilon$ and taking the limit $\epsilon \to 0$, find the matching condition determining the discontinuity in $\psi'(x)$ at x = 0. You can assume that $\psi(x)$ itself is everywhere continuous.
- **6.2** For $k^2 > 0$, solve the equation for x < 0 and x > 0 and then apply your matching condition, and the continuity of ψ at x = 0, to fix the coefficients of the solution for x < 0 in terms of those for x > 0. Use this to find a scattering solution, normalised so that the coefficient of e^{ikx} for x < 0 is equal to 1. What are the corresponding reflection and transmission coefficients?
- **6.3** The potential V(x) is now replaced by $W(x) = c\theta(x)$, where c > 0 is real and $\theta(x)$ is the Heaviside step function, equal to 1 for x > 0 and zero for $x \leq 0$. Derive the matching condition at x = 0 in this case, and find the general solution of the equation as in part **6.2**. For what possibly *c*-dependent values of $k^2 > 0$ does this new equation have a solution which tends to zero as $x \to +\infty$?



SECTION C

Q7 A real field u(x,t) is defined on the half-line $x \in (-\infty, 0]$, and has kinetic and potential energy

$$T = \int_{-\infty}^{0} dx \, \frac{1}{2} u_t^2 \,, \qquad V = \int_{-\infty}^{0} dx \left[\frac{1}{2} u_x^2 + e^u - e^\alpha \right] + B(u(0,t)) \,,$$

where B is a function of the value of the field at the x = 0 boundary, α is a real constant, and it is assumed that $u \ge \alpha$.

- **7.1** Which boundary conditions should be imposed on u, u_x and u_t as $x \to -\infty$ in order for the total energy to be finite?
- 7.2 Write down the action S. Use the principle of least (or stationary) action to derive the bulk equation of motion as well as the boundary condition for the field at x = 0.
- **7.3** Which choice of boundary potential energy B(u(0,t)) leads to Neumann (or free) boundary conditions?
- 7.4 It is given that

$$u(x,t) = \log\left(\frac{8f'(x+t)g'(x-t)}{(f(x+t)+g(x-t))^2}\right)$$

is a general solution of the bulk equation of motion, if f(y) and g(y) are smooth functions of one variable. Assuming that $g(y) = \epsilon f(y)$ and $f(-y) = \sigma f(y)$, where $\epsilon^2 = \sigma^2 = 1$, find the values of ϵ and σ for which the Neumann boundary condition is satisfied.