Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH41420-WE01

Title: Solitons V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 20%, Section B is worth 60%, and Section C is worth 20%. Within Sections A and B, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

SECTION A

Q1 Compute the dispersion relation for the equation

$$u_t + u_{xxx} + a u_{xxxx} + u_{xt} = 0 \ ,$$

where a is a real constant. For which values of a is there physical dissipation? In the case where there is no dissipation, physical or unphysical, compute the phase velocity and group velocity.

Q2 **2.1** Use the cyclic property of the trace to show that if a system is described by a matrix Lax pair equation

$$L_t = [M, L]$$

then the quantity $\mathrm{Tr}(L^2)$ is conserved, where Tr denotes the trace.

2.2 If L and M are given in terms of two functions $p(t)$ and $q(t)$ as

$$L = \begin{pmatrix} p & q \\ q & -p \end{pmatrix} \ , \qquad M = \begin{pmatrix} 0 & -q \\ q & 0 \end{pmatrix} \ ,$$

find the equations of motion for p and q that follow from the Lax pair equation. Calculate $\mathrm{Tr}(L^2)$, and use your equations of motion to show explicitly that it is conserved in this case.

SECTION B

Q3 The field $u = u(x,t)$ has energy

$$E = \int_{-\infty}^{+\infty} dx \, \left[\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + 2u^2(1-u^2)^2 \right] \, .$$

3.1 Use the Bogomol'nyi argument to find a lower bound for the energy in terms of the boundary values of the field u as $x \to \pm\infty$.

3.2 Now assume that $u \to 1$ as $x \to -\infty$ and $u \to 0$ as $x \to +\infty$. Find a numerical value for the lower bound of the energy. Write down the conditions that u must satisfy to saturate this bound, and find the function u that saturates the bound.

Q4 A field u satisfies the sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0$$

and has kinetic energy T and potential energy V given by

$$T = \int_{-\infty}^{+\infty} dx \, \frac{1}{2}u_t^2$$
$$V = \int_{-\infty}^{+\infty} dx \, \left(\frac{1}{2}u_x^2 + 1 - \cos u \right) \, .$$

4.1 Which boundary conditions should u obey to ensure that the total energy $E = T + V$ be finite?

4.2 Show that with these boundary conditions the total energy is conserved.

4.3 It is given that

$$u(x,t) = 4 \arctan \left[\cot\varphi \cdot \frac{\sin(t \sin\varphi)}{\cosh(x\cos\varphi)} \right] \, ,$$

is a solution of the sine-Gordon equation, where $\varphi \in (0, 2\pi)$ is a constant. Calculate the kinetic energy T and the potential energy V at times $t = n\tau/4$, where $\tau = 2\pi/\sin\varphi$ and $n = 0, 1, 2, 3, 4$. You may use without proof the identities

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2} \, , \quad \frac{d}{dx} \tanh x = \frac{1}{\cosh^2 x} = \text{sech}^2 x \, .$$

Q5 **5.1** If M is a differential operator acting on functions which decay as $x \to \pm\infty$, define $M^\dagger$, the adjoint of M, with respect to the inner product

$$(\phi, \chi) = \int_{-\infty}^{\infty} \phi^*(x)\chi(x)\, dx\,.$$

What does it mean for M to be (a) symmetric (self-adjoint) or (b) antisymmetric (skew) with respect to this inner product?

5.2 If N is a second differential operator also acting on functions which decay as $x \to \pm\infty$, show that $(MN)^\dagger = N^\dagger M^\dagger$.

5.3 Let $D = d/dx$ and $u(x)$ be a real function decaying as $x \to \pm\infty$. Classify each of the following differential operators as either symmetric, antisymmetric, or neither, giving reasons in each case:

$$B_1 = D\,, \quad B_2 = u\,, \quad B_3 = uD\,, \quad B_4 = uD + Du\,, \quad L = D^2 + u\,.$$

5.4 If B is another differential operator and $[L, B]$ is multiplicative (and real), with $L = D^2 + u$ as above, show that L commutes with the symmetric part of B. Explain briefly why this fact might be useful in searching for the equations of the KdV hierarchy.

Q6 Consider the Schrödinger equation

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi(x) = k^2\psi(x)$$

where $V(x) = -b\delta(x)$, b is real, and $\delta(x)$ is the Dirac delta function.

6.1 By integrating the equation from $-\epsilon$ to $+\epsilon$ and taking the limit $\epsilon \to 0$, find the matching condition determining the discontinuity in $\psi'(x)$ at $x = 0$. You can assume that $\psi(x)$ itself is everywhere continuous.

6.2 For $k^2 > 0$, solve the equation for $x < 0$ and $x > 0$ and then apply your matching condition, and the continuity of ψ at $x = 0$, to fix the coefficients of the solution for $x < 0$ in terms of those for $x > 0$. Use this to find a scattering solution, normalised so that the coefficient of e^{ikx} for $x < 0$ is equal to 1. What are the corresponding reflection and transmission coefficients?

6.3 The potential $V(x)$ is now replaced by $W(x) = c\theta(x)$, where $c > 0$ is real and $\theta(x)$ is the Heaviside step function, equal to 1 for $x > 0$ and zero for $x \leq 0$. Derive the matching condition at $x = 0$ in this case, and find the general solution of the equation as in part **6.2**. For what possibly c-dependent values of $k^2 > 0$ does this new equation have a solution which tends to zero as $x \to +\infty$?

SECTION C

Q7 A real field $u(x,t)$ is defined on the half-line $x \in (-\infty, 0]$, and has kinetic and potential energy

$$T = \int_{-\infty}^{0} dx \; \frac{1}{2} u_t^2 \; , \qquad V = \int_{-\infty}^{0} dx \left[\frac{1}{2} u_x^2 + e^u - e^\alpha \right] + B(u(0,t)) \; ,$$

where B is a function of the value of the field at the $x = 0$ boundary, α is a real constant, and it is assumed that $u \geq \alpha$.

7.1 Which boundary conditions should be imposed on u, u_x and u_t as $x \to -\infty$ in order for the total energy to be finite?

7.2 Write down the action S. Use the principle of least (or stationary) action to derive the bulk equation of motion as well as the boundary condition for the field at $x = 0$.

7.3 Which choice of boundary potential energy $B(u(0,t))$ leads to Neumann (or free) boundary conditions?

7.4 It is given that

$$u(x,t) = \log \left(\frac{8 f'(x+t) g'(x-t)}{(f(x+t) + g(x-t))^2} \right)$$

is a general solution of the bulk equation of motion, if $f(y)$ and $g(y)$ are smooth functions of one variable. Assuming that $g(y) = \epsilon f(y)$ and $f(-y) = \sigma f(y)$, where $\epsilon^2 = \sigma^2 = 1$, find the values of ϵ and σ for which the Neumann boundary condition is satisfied.