Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH4151-WE01

Title:
Topics in Algebra and Geometry IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 For $\tau \in \mathbb{H}$, the upper half plane, and $z \in \mathbb{C}$, define

$$f(z;\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n z}.$$

Show in detail that $f(z;\tau)$ is a holomorphic function in both z and τ.

Q2 (i) With $f(z;\tau)$ as in Question 1 show that

$$\begin{aligned} f(z+1;\tau) &= f(z;\tau); \\ f(z+\tau;\tau) &= e^{-\pi i\tau - 2\pi i z} f(z;\tau). \end{aligned}$$

(ii) Let $a_1, \ldots, a_k; b_1, \ldots, b_k \in \mathbb{C}$ such that $\sum_{i=1}^k a_i - b_i \in \mathbb{Z}$. Show that

$$F(z) = \prod_{i=1}^k \frac{f(z-a_i;\tau)}{f(z-b_i;\tau)}$$

is an elliptic function with respect to $\Omega = \mathbb{Z}\tau \oplus \mathbb{Z}$.

Q3 Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

(a) Give a basis of the complex vector space $M_{24}(\Gamma)$. Do the same for $S_{24}(\Gamma)$.

(b) Let $f \in M_{12}(\Gamma)$ and assume that $f(\tau) = \sum_{n=0}^\infty a_n q^n$, with $a_n \in \mathbb{Q}$ for all n.

(i) Show that there exist $a, b \in \mathbb{Q}$ such that $f(\tau) = a\,\Delta(\tau) + b\,E_{12}(\tau)$.

(ii) Show that there exists a $\lambda \in \mathbb{N}$ such that $\lambda \cdot a_n \in \mathbb{Z}$ for all n.

Q4 (a) State the $k/12$ -formula and define each term.

(b) Let $0 \neq g(\tau) \in M_{10}(\Gamma)$ with $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Show that $g(i) = g(\omega) = 0$, where $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Show further that $g(\frac{1}{3} + 3i) \neq 0$.

SECTION B

Q5 Let $\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$ and let $\wp(z)$ denote the associated Weierstrass $\wp$-function.

(i) Show that $f(z) = \wp(2z) - \wp(z)$ is an elliptic function for Ω. What is the order of f?

(ii) Explicitly find all zeros of f and draw them in the standard fundamental domain for Ω.

(iii) Explain why $f(z)$ can be written as a rational function in $\wp(z)$. Find the denominator of degree 3 for this rational function and express it using the Eisenstein series g_2 and g_3. What is the degree of the numerator? Find its leading term.

Q6 (i) Consider the Weierstrass ζ-function for a lattice Ω in $\mathbb{C}$. Show that there exists a function η on Ω such that $\zeta(z+\omega) = \zeta(z) + \eta(\omega)$ for all $z \in \mathbb{C}$ and $\omega \in \Omega$. (You may assume the relationship of ζ to $\wp$).

(ii) Let b_1, b_2, a_1 be three pairwise Ω-inequivalent points in $\mathbb{C}$. Use $\zeta(z)$ to explicitly construct (with proof) a non-zero elliptic function f which

(1) vanishes at a_1 of order at least 1; and

(2) has simple poles at b_1 and b_2 and no poles anywhere else $(\bmod\ \Omega)$.

Discuss the uniqueness of an elliptic function satisfying properties (1) and (2).

(iii) There must be another (typically) inequivalent zero a_2 of f (Why?). Give an explicit formula for a_2 in terms of a_1, b_1, b_2 and a criterion when the zero of f at a_1 is actually a double zero.

Q7 Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and let $\Delta(\tau) \in S_{12}(\Gamma)$ denote the discriminant function.

(a) (i) We define $g(\tau) = \Delta(2\tau)\cdot\Delta(\tau/2)\cdot\Delta\left(\frac{\tau+1}{2}\right)$. Show that $g(\tau) \in S_{36}(\Gamma)$.

(ii) Show that $g(\tau) = -\Delta^3(\tau)$.

(b) Let $0 \neq f(\tau) \in M_k(\Gamma)$ for some even integer $k \geq 12$. Show that there exists a constant $\alpha \in \mathbb{C}$ such that

$$f(\tau) = \alpha\, E_4(\tau)^{ord_\omega(f)} E_6(\tau)^{ord_i(f)} \Delta(\tau)^{ord_\infty(f)+\gamma(f)} \prod_{P\in\Gamma\backslash\mathbb{H}, P\neq i,\omega} (j(\tau)-j(P))^{ord_P(f)},$$

where $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, and $\gamma(f) := \sum_{P\in\Gamma\backslash\mathbb{H}, P\neq i,\omega} ord_P(f)$. Here $j(\tau)$ denotes the j-function.

Q8 Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and let $0 \neq f(\tau) = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma)$ for some $k \in \mathbb{N}$.

(a) Show that the function $\widetilde{f}(\tau) := \mathrm{Im}(\tau)^{k/2}|f(\tau)|$ is bounded on $\mathbb{H}$. Conclude that there exists a $w \in \mathbb{H}$ such that $\widetilde{f}(\tau) \leq \widetilde{f}(w)$ for all $\tau \in \mathbb{H}$.

(b) Let p be a prime number and let T_p be the p^{th} Hecke operator. Assume that $T_p f = \lambda f$ for some $\lambda \in \mathbb{C}$. Show that

$$|\lambda| \leq p^{k/2}\left(1 + \frac{1}{p}\right).$$

(Hint: Consider $\widetilde{T_p f}(w)$ and recall that $T_p f(\tau) = p^{k-1} f(p\tau) + \frac{1}{p}\sum_{j=0}^{p-1} f\left(\frac{\tau+j}{p}\right)$ for all $\tau \in \mathbb{H}$.)

(c) With notation as above we now set $g(\tau) := f(\tau) - \lambda f(p\tau) + p^{k-1} f(p^2\tau)$. We write $g(\tau) = \sum_{m=1}^{\infty} b_m q^m$. Show that

$$b_m = \begin{cases} a_m, & \text{if } \gcd(p, m) = 1, \\ 0, & \text{otherwise.} \end{cases}$$