Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH4161-WE01

Title:

Algebraic Topology IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 (a) State the Mayer-Vietoris long exact sequence.

(b) Compute the homology of the figure eight, i.e. the wedge $S^1 \vee S^1$ of two circles, using the Mayer-Vietoris sequence.

Q2 True or false? In each case either give a proof or a counterexample, as appropriate.

(a) Suppose that $X = \mathring{A} \cup \mathring{B}$ where $A, B \subseteq X$ are such that $A \simeq S^1$, $B \simeq S^1$, and $A \cap B \simeq S^1$. Then $X \simeq S^1$.

(b) Let $f\colon X \to Y$ be a surjective map between spaces. Then for all $n \in \mathbb{N}_0$, the induced map on homology $f_*\colon H_n(X) \to H_n(Y)$ is a surjective homomorphism.

(c) If $0 \to \mathbb{Z}/2 \to G \to \mathbb{Z}/3 \to 0$ is a short exact sequence of abelian groups, then $G \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3$.

Q3 It is a fact, which you may use, that every compact 3-manifold admits a CW structure with finitely many cells, and these cells are of dimension at most three.

(a) Let M be a closed, orientable 3-manifold. Prove that the Euler characteristic vanishes, that is $\chi(M) = 0$.

(b) Let N be a compact, orientable 3-manifold with boundary homeomorphic to the closed, orientable surface Σ_g of genus g. What is $\chi(N)$? You may assume without proof that $\chi(\Sigma_g) = 2 - 2g$.

Q4 (a) Let (X, A) be a pair of spaces. Define the relative homology groups $H_i(X, A)$, for $i \in \mathbb{N}_0$.

(b) Let X be a path connected space and let $x \in X$. Prove that $H_0(X, x) = 0$ directly from the definitions. If you want to use a result from lectures then you must prove it from scratch.

SECTION B

Q5 Consider the figure eight $S^1 \vee S^1$ embedded in $\mathbb{R}^3$, for example as the subspace

$$\infty := \Big(\big\{(x, y, 0) \in \mathbb{R}^3 \mid (x-1)^2 + y^2 = 1\big\} \cup \big\{(x, y, 0) \in \mathbb{R}^2 \mid (x+1)^2 + y^2 = 1\big\}\Big) \subseteq \mathbb{R}^3.$$

Compute the homology groups of the complement $\mathbb{R}^3 \setminus \infty$.

You may assume without proof your knowledge of the homology groups of the circle S^1, the figure eight $S^1 \vee S^1$, and the torus $S^1 \times S^1$, provided you state your assumed knowledge clearly.

Q6 (a) For each integer $k \geq 0$, decide whether or not there exists a continuous map $S^3 \to \mathbb{RP}^3$ of degree k, and justify your decision.

(b) In each of the following cases, decide whether or not there exists a closed, connected 4-manifold M whose only nonzero reduced homology groups are:

(i) $\widetilde{H}_2(M) \cong \mathbb{Z}^3$ and $\widetilde{H}_4(M) \cong \mathbb{Z}$;

(ii) $\widetilde{H}_2(M) \cong \mathbb{Z}^3 \oplus \mathbb{Z}/3$ and $\widetilde{H}_4(M) \cong \mathbb{Z}$;

(iii) $\widetilde{H}_1(M) \cong \mathbb{Z}/2 \cong \widetilde{H}_3(M)$ and $\widetilde{H}_4(M) \cong \mathbb{Z}$;

(iv) $\widetilde{H}_1(M) \cong \mathbb{Z}/2 \cong \widetilde{H}_3(M)$.

Either describe a closed 4-manifold with the given reduced homology groups (you do not have to prove rigorously that the homology is as claimed), or prove that no such manifold exists.

Q7 We can write a CW decomposition for the complex projective plane as

$$\mathbb{CP}^2 = e^0 \cup e^2 \cup_\eta e^4 = S^2 \cup_\eta e^4,$$

where $\eta\colon S^3 \to S^2$ is the attaching map for the 4-cell.

(a) Compute the cohomology groups of $\mathbb{CP}^2$ and $S^2 \vee S^4$ using CW cohomology, or otherwise.

(b) Suppose that η is homotopic to a constant map $c\colon S^3 \to S^2$. Write down a homotopy equivalence $f\colon \mathbb{CP}^2 \to S^2 \vee S^4$. You do not have to prove that f is a homotopy equivalence.

(c) Prove that the cup product map

$$- \smile -\colon H^2(S^2 \vee S^4; \mathbb{Z}) \times H^2(S^2 \vee S^4; \mathbb{Z}) \to H^4(S^2 \vee S^4; \mathbb{Z})$$

is the zero map.

(d) Prove that the cup product map

$$- \smile -\colon H^2(\mathbb{CP}^2; \mathbb{Z}) \times H^2(\mathbb{CP}^2; \mathbb{Z}) \to H^4(\mathbb{CP}^2; \mathbb{Z})$$

is not the zero map.

(e) Deduce that η is not homotopic to a constant map.

Q8 (a) State and prove the Brouwer fixed point theorem.

(b) Let $X := \{x \in \mathbb{R}^2 \mid \|x\| < 1\}$ be the open unit disc. Is there a continuous map $f\colon X \to X$ without a fixed point?