EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH4171-WE01

Title:
Riemannian Geometry IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 - y^2 - 2z^2 = 1\}$ be a hyperboloid.

1.1 Show that M is a 2-dimensional smooth manifold.

1.2 Find two curves $\gamma_i \colon [0,1] \to M$, $i = 1, 2$, such that $\gamma_1(0) = \gamma_2(0) = (1, 0, 0)$ and $\{\gamma_1'(0), \gamma_2'(0)\}$ is a basis for $T_{(1,0,0)}M$.

Q2 Let X, Y be two differentiable vector fields on $\mathbb{R}^3$ defined by

$$X(x, y, z) = 2y\frac{\partial}{\partial x} + (z - 2x)\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},$$

$$Y(x, y, z) = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}.$$

2.1 Compute the Lie bracket $[X, Y]$.

2.2 Let $r > 0$ and $S^2(r) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = r^2\}$ be the 2-dimensional sphere of radius r. Show that the restriction of $[X, Y]$ to $S^2(r)$ is a vector field on $S^2(r)$.

Q3 **3.1** Define the Levi–Civita connection on a Riemannian manifold.

3.2 Let (M, g) be a Riemannian manifold and let R be its curvature tensor. Prove *Bianchi's first identity*

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0,$$

assuming that *Jacobi's identity*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

holds for $X, Y, Z \in \mathcal{X}(M)$. You do not need to prove Jacobi's identity.

Q4 **4.1** State the symmetry properties of the curvature tensor R of a Riemannian manifold (M, g).

4.2 Prove or disprove the following statement: *The smooth manifold $\mathbb{R}^n$ admits a complete Riemannian metric with sectional curvature $K \geq 1$.*

4.3 Let (M, g) be a Riemannian manifold and let $c \colon [a, b] \to M$ be a geodesic. Let $X \in \mathcal{X}_c(M)$ be a parallel vector field along c. Show that the function $f \colon [a, b] \to \mathbb{R}$ given by $t \mapsto \langle X(t), c'(t)\rangle$ is constant.

SECTION B

Q5 Let $M(n, \mathbb{R})$ denote the set of all $n \times n$ matrices with real entries, let

$$\mathrm{Sym}(n) = \{\, A \in M(n, \mathbb{R}) \mid A^T = A \,\}$$

be the set of $n \times n$ symmetric matrices, and let

$$J = \begin{pmatrix} \mathrm{Id}_{n-1} & 0 \\ 0 & -1 \end{pmatrix},$$

where Id_k denotes the $k \times k$ identity matrix in $M(k, \mathbb{R})$. Let

$$\mathrm{O}(n-1,1) = \{ A \in M(n, \mathbb{R}) \mid AJA^T = J \}.$$

You may assume, without proof, that $M(n, \mathbb{R})$ and $\mathrm{Sym}(n)$ are Lie groups (with respect to matrix multiplication).

5.1 Let $f\colon M(n, \mathbb{R}) \to \mathrm{Sym}(n)$ be given by $f(A) = AJA^T$. Show that, given $A, B \in M(n, \mathbb{R})$,

$$Df_A(B) = AJB^T + BJA^T,$$

and conclude that, for $A \in \mathrm{O}(n-1,1)$ and $C \in \mathrm{Sym}(n)$,

$$Df_A(\frac{1}{2}CJA) = C.$$

You may assume, without proof, that every matrix in $\mathrm{O}(n-1,1)$ is invertible and that $\mathrm{O}(n-1,1)$ is a group under matrix multiplication.

5.2 Show that $J \in \mathrm{Sym}(n)$ is a regular value of f.

5.3 Show that $\mathrm{O}(n-1,1)$ is a differentiable manifold and determine its dimension.

Q6 Let M and N be smooth manifolds. A map $f\colon M \to N$ is a *local diffeomorphism* if every point $p \in M$ has a neighborhood U such that $f(U)$ is open in N and $f|_U\colon U \to f(U)$ is a diffeomorphism.

6.1 Show that the differential Df_p of a local diffeomorphism $f\colon M \to N$ is a linear isomorphism of vector spaces for any $p \in M$.

6.2 State the definition of an oriented manifold.

6.3 Suppose now that the manifolds M and N are connected and oriented. Assume, without proof, that the given orientations induce orientations on each tangent space of M and N via the coordinate tangent vectors. A local diffeomorphism $f\colon M \to N$ is *orientation-preserving* if, for each $p \in M$, the isomorphism $Df_p\colon T_pM \to T_{f(p)}N$ preserves orientation, and *orientation-reversing* if $Df_p\colon T_pM \to T_{f(p)}N$ reverses orientation. Show that the composition of two orientation-preserving local diffeomorphisms is orientation-preserving.

Q7 Let (M, g) be a Riemannian manifold with curvature tensor R.

7.1 Show that

$$R(X,Y)(fZ) = fR(X,Y)Z$$

for $X, Y, Z \in \mathcal{X}(M)$ and $f \in C^\infty(M)$.

7.2 Suppose that there exists a constant $C \in \mathbb{R}$ such that

$$\langle R(X,Y)W, Z\rangle = C(\langle X, Z\rangle)\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle). \qquad (*)$$

Show that (M, g) is an Einstein manifold, i.e., $\mathrm{Ric} = \lambda g$ for some function $\lambda \in C^\infty(M)$, and determine the concrete value of λ.

7.3 Let (M, g) be a Riemannian manifold satisfying equation $(*)$ for a constant $C \leq 0$. Let $c\colon [a, b] \to M$ be a geodesic parametrised by arc-length and $X \in \mathcal{X}_c(M)$ be a parallel vector field along c with $\langle X(t), c'(t)\rangle = 0$ for all $t \in [a, b]$. Show that the vector field

$$J(t) = \cosh(\sqrt{-C}t)X(t)$$

is a Jacobi field along c.

Q8 **8.1** Let (M, g) be a Riemannian manifold and let ∇ be its Levi–Civita connection. Let X, Y be smooth vector fields on M and suppose that Y has constant length, i.e., $\|Y(p)\|$ is constant for all $p \in M$. Show that $(\nabla_X Y)(p)$ is orthogonal to $Y(p)$ for all $p \in M$.

8.2 State the definition of a tensor field on a smooth manifold M.

8.3 Let M be a smooth manifold, let ∇ be a linear connection on M, and let F be a $\binom{1}{1}$ tensor field on M. Show that the map $\nabla F\colon \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ given by

$$\nabla F(X, Y) = \nabla_X(F(Y)) - F(\nabla_X Y)$$

defines a $\binom{2}{1}$ tensor field on M.