Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH4181-WE01

Title: Mathematical Finance IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 20%, Section B is worth 60%, and Section C is worth 20%. Within Sections A and B, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 (a) State the Cox–Ross–Rubinstein formula for the price at time $T - t$ of a European call option with strike price K and expiry date T. Explain clearly any notation you use in your formula.

(b) Using your formula in part (a), calculate the price at time 0 of a European call option with strike price $K = 40$ and expiry date $T = 5$ on an underlying risky asset with price S_t that evolves according to the 5-period binomial model with $S_0 = 40, u = 1.2, d = 0.8, p_u = 0.8, p_d = 0.2$ and where the interest rate is compounded discretely at rate $r = 0.1$ per time period.

(c) Explain what happens to the Cox–Ross–Rubinstein price in part (b) for values of $K < 13$.

Q2 Under the Black–Scholes model, let $C(K, T, \sigma, S_0, r)$ be the price of a European call option with strike price K and time to expiry T where the underlying stock has volatility σ and current price S_0, and r is the risk-free rate.

(a) State the Black-Scholes formula for $C(K, T, \sigma, S_0, r)$.

(b) Find $\dfrac{\partial C}{\partial \sigma}$ and show that the price of a European call option, as a function of the volatility parameter $\sigma > 0$, is strictly increasing.

(c) Does the price of a European put option increase with $\sigma > 0$ as well?

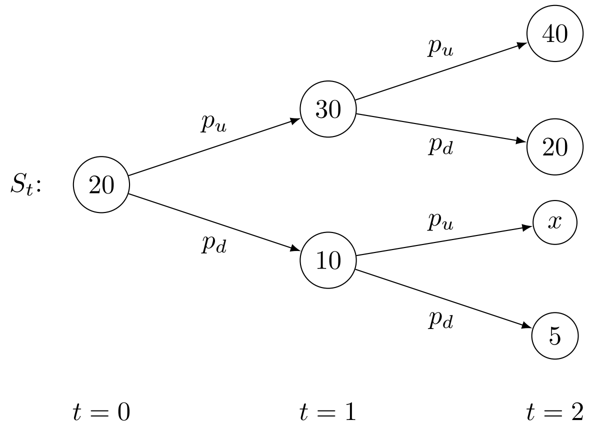
SECTION B

Q3 (a) Define the four main types of barrier option derived from a European-style option with contract function Φ and expiry date T. In each case provide a random variable that represents the random payoff of the barrier option. You may give your answer in terms of the price S_t of the underlying risky asset, and the contract function Φ, but you should define any other notation you use.

(b) Suppose that the price S_t of a risky asset evolves according to the 3-period binomial model, with parameters $S_0 = 216, u = 7/6, d = 5/6, p_u = 1/4, p_d = 3/4$ and that interest is compounded at rate $r = 1/12$ per time period.

Calculate the prices at all times $t = 0, 1, 2, 3$ of a down-and-out barrier (European) put option with strike price $K = 250$, barrier 160, and expiry date $T = 3$.

Q4 Consider the market (B_t, S_t) where $B_t = (5/4)^t$ for $t = 0, 1, 2$, and S_t evolves randomly according to the diagram below, with $p_u = p_d = 1/2$ at each node.



(a) State the conditions on x that guarantee that the market is arbitrage free and calculate the risk-neutral probabilities at every node in this case.

(b) Give the definition of an American put option with strike price K and expiry time T and describe the algorithm for pricing an American put option on this market. (You do not need to prove the correctness of the algorithm.)

(c) Calculate the price at time 0 of an American put option with strike price $K = 20$ and expiry date $T = 2$, as a function of x.

(d) Now suppose that the strike price of the option is $K = 30$. Prove that it is always optimal to exercise at $t = 0$, whatever the value of x (assuming the market is arbitrage free).

Q5 **5.1** Show that if $W = (W_t, t \geq 0)$ is a Brownian motion and $c > 0$, then the process $(c^{-1/2}W_{ct}, t \geq 0)$ is also a Brownian motion.

5.2 Let us use the notation $(x)_+ = \max(x, 0)$ for the rest of this question. Consider

$$H(r, \sigma, T) := \mathbb{E}\left[\left(10 - \max_{0 \leq t \leq T} S_t\right)_+\right]$$

where the underlying stock price evolves as

$$dS_t = rS_t dt + \sigma S_t dW_t, \qquad S_0 = 1$$

where $r, \sigma > 0$ are some constant parameters. Find two functions

$$\tilde{r}(c, r, \sigma) \qquad \text{and} \qquad \tilde{\sigma}(c, r, \sigma)$$

such that for any $c > 0$

$$H(r, \sigma, cT) = H(\tilde{r}, \tilde{\sigma}, T).$$

5.3 Show that if

$$\alpha := \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right) > 0,$$

then

$$\mathbb{E}\left[\left(10 - \max_{0 \leq t \leq T} S_t^{1/2}\right)_+\right] \leq 10^{\alpha/\sigma} e^{-\frac{3}{8}\alpha^2 T} H(r, \sigma, T/4).$$

Q6 Consider a portfolio $(P_t, t \geq 0)$ described by the stochastic differential equation

$$\frac{dP_t}{P_t} = a_t \frac{dS_t^{(1)}}{S_t^{(1)}} + (1 - a_t)\frac{dS_t^{(2)}}{S_t^{(2)}}, \qquad P_0 = 1$$

where $(S^{(1)}, S^{(2)})$ are two assets that evolve according to

$$\frac{dS_t^{(i)}}{S_t^{(i)}} = \mu_i dt + \sigma_i dW_t^{(i)}, \qquad S_0^{(i)} = 1 \qquad i = 1, 2$$

with $(W_t^{(1)})_{t\geq 0}$ and $(W_t^{(2)})_{t\geq 0}$ being two independent Brownian motions. Here $\mu_1, \mu_2, \sigma_1, \sigma_2 > 0$ are some constant parameters, and $(a_t, t \geq 0)$ is a continuous process adapted to $\mathcal{F}_t := \sigma((W_u^{(1)}, W_u^{(2)}), u \in [0, t])$ with the property that $a_t \in [0, 1]$ for all $t \geq 0$.

6.1 Let $R_t = \log P_t$. Express $\mathbb{E}[R_t]$ in terms of $(a_s, s \geq 0)$ and $(\mu_1, \mu_2, \sigma_1, \sigma_2)$.

6.2 Show that there exists some constant $C > 0$ independent of t and $(a_s, s \geq 0)$ such that

$$\mathbb{E}[[R]_t] \leq Ct \qquad \forall t \geq 0.$$

6.3 Fix some $\alpha > 0$ and consider

$$a_t := \frac{(S_t^{(1)})^\alpha}{(S_t^{(1)})^\alpha + (S_t^{(2)})^\alpha}.$$

By finding the distribution of $S_t^{(1)}/S_t^{(2)}$ or otherwise, show that for any $\epsilon \in (0, 1)$,

$$\lim_{t\to\infty} \mathbb{P}(a_t \in [\epsilon, 1 - \epsilon]) = 0.$$

6.4 Let $m := \max_{i=1,2}\{\mu_i - \frac{1}{2}\sigma_i^2\}$. Using a_t defined in the previous part, show that

$$\lim_{t\to\infty} t^{-1}\mathbb{E}[R_t] = m \qquad \text{and} \qquad \lim_{t\to\infty} t^{-2}\text{Var}(R_t) = 0$$

and conclude that R_t/t converges in probability to m as $t \to \infty$.

SECTION C

Q7 (a) Write down the usual unbiased estimators for the mean and variance of a random variable X constructed from the sample $X_1, X_2, \ldots, X_M$. Your answer should make clear what conditions on the random variables $X_1, X_2, \ldots, X_M$ are required to ensure the estimators are unbiased.

(b) Suppose the share price S_t of a risky asset evolves randomly according a geometric Brownian motion with drift μ and volatility σ, and that interest is compounded continuously at rate r. Describe in detail the Monte Carlo algorithm that produces an approximate 95% confidence interval for the no-arbitrage price at time 0 of a European call option with strike price K and expiry date T from a sample of M standard Normal random variables.

(c) Consider an Asian call option on the same risky asset whose payoff at time T equals $\max(A_S - K, 0)$ where $A_S = \frac{1}{n}\sum_{i=1}^{n} S_{T_i}$ for some fixed times $0 < T_1 < T_2 < \cdots < T_n = T$. Explain how you would modify your Monte Carlo algorithm from part (b) to price this Asian call option, using a sample of $M \times n$ standard Normal random variables.

[Hint: for any sequence of times $0 = T_0 < T_1 < \cdots < T_n = T$ the random variables $\log(S_{T_i}/S_{T_{i-1}}), i = 1, \ldots, n$, are independent Normals.]