EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH41820-WE01

Title:

Fluid Mechanics V

Time:	3 hours	
Additional Material provided:	Formula sheet.	
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 20%, Section B is worth 60%, and Section C is worth 20%. Within Sections A and B, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 The linearised equations for small-amplitude water waves on the surface of a domain $D_t = \{(x, z) : -h < z < \zeta(x, t), -\infty < x < \infty\}$ are given by

$$\Delta\phi = 0 \quad \text{in } -h < z < 0,$$

$$\frac{\partial \phi}{\partial t} + g\zeta = 0 \quad \text{at } z = 0,$$

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{at } z = 0,$$

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = -h.$$

(a) Aside from their small amplitude and the ideal nature of the fluid, what two other main assumptions have been made about the flow in deriving the above equations?

(b) Assuming a solution of the form $\phi(x, z, t) = X(x)Z(z)\sin(\omega t)$, find the most general solutions for $X(x)$ and $Z(z)$ that satisfy these equations.

(c) Find the dispersion relation. Are these waves dispersive?

Q2 A fixed volume V is filled with an unforced ideal barotropic fluid obeying the relation $P(\rho) = \rho^\gamma$ with γ constant.

(a) Write down the continuity and momentum equations for this fluid in terms of $\boldsymbol{u}$ and ρ only (not p).

(b) Assuming that $\boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S} = \boldsymbol{0}$ on the boundary ∂V, use the equations in (a) to show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \frac{1}{2}\rho|\boldsymbol{u}|^2 \,\mathrm{d}V = -\int_V \gamma\rho^{\gamma-1}\boldsymbol{u} \cdot \nabla\rho \,\mathrm{d}V.$$

SECTION B

Q3 (a) Write down the Lagrangian form of the vorticity equation for an ideal, incompressible fluid.

(b) Starting from the ansatz that vorticity can be expressed in index notation as

$$\omega_i = C_j \frac{\partial x_i}{\partial a_j},$$

where $\boldsymbol{C}(\mathbf{x}, t)$ is some unknown vector field, show that the solution of the Lagrangian form of the vorticity equation can be written as

$$\omega_i(\mathbf{x}, t) = \frac{\partial x_i}{\partial a_j}\omega_j(\mathbf{a}, 0).$$

(c) Consider a fluid flow with vorticity at $t = 0$ given by

$$\boldsymbol{\omega}(\mathbf{x}, 0) = \begin{cases} \Omega\mathbf{e}_z & x^2 + y^2 \leq R_0^2 \\ 0 & x^2 + y^2 > R_0^2, \end{cases}$$

where $\Omega, R_0 \in \mathbb{R}$. Find $\boldsymbol{\omega}(\mathbf{x}, t)$, the vorticity at later times, when this flow is evolved by a potential flow of the form

$$\mathbf{u} = -x\mathbf{e}_x - y\mathbf{e}_y + 2z\mathbf{e}_z.$$

Q4 Consider an ideal fluid flow in the domain between two spheres of radius a and b, where $b > a > 0$. Working in *cylindrical* coordinates the flow is described by the Stokes' stream function $\psi(r, z)$ such that

$$\mathbf{u}(r, z) = -\frac{1}{r}\frac{\partial \psi}{\partial z}\mathbf{e}_r + \frac{1}{r}\frac{\partial \psi}{\partial r}\mathbf{e}_z.$$

(a) Show that the vorticity is given by

$$\boldsymbol{\omega} = -\frac{1}{r}L\psi \mathbf{e}_\theta,$$

where $L\psi$ is an operator applied to ψ you must determine.

(b) The vorticity between the spheres is such that

$$L\psi = \begin{cases} \Omega r^2 & a \leq \rho \leq b \\ 0 & \text{otherwise} \end{cases},$$

where $\rho = \sqrt{r^2 + z^2}$ and $\Omega \in \mathbb{R}$. Assuming that the potential takes the form $\psi(r, z) = r^2 f(\rho^2)$, show that in the region $a \leq \rho \leq b$, f satisfies

$$10f'(t) + 4tf''(t) = \Omega,$$

where $t = \rho^2$, and solve to find the general expression for $f(t)$.

(c) Assume that $f(\rho = a) = f(\rho = b) = 0$ and find f, expressing your answer in terms of ρ. With these boundary conditions how does the flow behave at $\rho = a$ and b and why?

Q5 A bugle is modelled as an infinitely-long straight tube $0 < x < \infty$ with varying cross-sectional area $A(x) = \mathrm{e}^{ax}$ for some constant a. We propose to model the air inside the bugle by a one-dimensional flow $\boldsymbol{u} = u(x, t)\boldsymbol{e}_x$, $p = p(x, t)$, $\rho = \rho(x, t)$. For consistency, conservation of mass implies that the continuity equation must be modified to

$$A\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(A\rho u) = 0.$$

(a) Assuming that they are unmodified, write down the other unforced compressible barotropic Euler equations for this flow.

(b) Assume a basic state with uniform density ρ_0, uniform pressure p_0 and zero velocity. Replacing $\rho \to \rho + \rho_0$ and $p \to p + p_0$ in your equations from (a), derive the resulting linearised equations for the perturbations ρ, u and p.

(c) Show that the pressure perturbation p satisfies the modified wave equation

$$\frac{\partial^2 p}{\partial t^2} = c_0^2\left(\frac{\partial^2 p}{\partial x^2} + a\frac{\partial p}{\partial x}\right).$$

(d) By taking the ansatz $p(x, t) = X(x)\mathrm{e}^{i\omega t}$, find the condition on ω for the existence of travelling waves that move along the tube. How does the amplitude of these waves depend on A?

Q6 An incompressible axisymmetric flow in the infinite cylinder $a < r < b$ has the form

$$u_r(r,z,t)\boldsymbol{e}_r + \Big[U(r) + u_\theta(r,z,t)\Big]\boldsymbol{e}_\theta + u_z(r,z,t)\boldsymbol{e}_z, \qquad p_0(r) + p(r,z,t), \qquad \rho_0,$$

where u_r, u_θ, u_ϕ and p represent perturbations around a steady flow $U(r)\boldsymbol{e}_\theta$ with pressure $p_0(r)$ and uniform density ρ_0.

(a) Write out the four unforced incompressible Euler equations satisfied by u_r, $U + u_\theta$, u_z and $p_0 + p$. What boundary conditions are required on $r = a$ and $r = b$?

(b) What condition on U and p_0 is required in order that they satisfy the steady Euler equations when there is no perturbation?

(c) Hence write down the linearised (unforced incompressible Euler) equations satisfied by the perturbations u_r, u_θ, u_z, and p.

(d) By assuming the ansatzes

$$u_r = \hat{u}_r(r)\mathrm{e}^{i(kz-\omega t)},\ u_\theta = \hat{u}_\theta(r)\mathrm{e}^{i(kz-\omega t)},\ u_z = \hat{u}_z(r)\mathrm{e}^{i(kz-\omega t)},\ p = \hat{p}(r)\mathrm{e}^{i(kz-\omega t)},$$

with $k \in \mathbb{R}$ and $\omega \in \mathbb{C}$, show that the linearised equations reduce to the single ODE

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r\hat{u}_r\Big)\right] - k^2\hat{u}_r = -\frac{k^2}{\omega^2}\Phi(r)\hat{u}_r,$$

where $\Phi(r) = \dfrac{1}{r^3}\dfrac{\mathrm{d}}{\mathrm{d}r}\Big(r^2U^2\Big)$.

SECTION C

Q7 Water of a constant depth H lies at rest in the region $x > 0$ adjacent to a wall at $x = 0$. At $t = 0$ the wall begins to move with a speed $U = \alpha t$ in the positive x direction, so that water begins to pile up in front of it. We propose to model this with the shallow water equations:

$$\partial_t v + v\partial_x v + g\partial_x h = 0,$$
$$\partial_t h + v\partial_x h + h\partial_x v = 0.$$

(a) Show that the Riemann invariants $I_\pm = v \pm 2c$, where $c = \sqrt{gh}$, are constant along the characteristics satisfying $dx_\pm/dt = v \pm c$.

(b) In the region $x > c_0 t$, where $c_0 = \sqrt{gH}$, show that both characteristics are straight lines.

(c) Consider a point $P = (x_1, t_1)$ in the (x,t) plane that sits on the moving wall. The x_- characteristic that passes through P originates in the region $x > c_0 t$. Show that

$$v(x_1, t_1) = \alpha t_1, \quad c(x_1, t_1) = \frac{1}{2}\alpha t_1 + c_0.$$

Hence show that the x_+ characteristic passing through P is a straight line and find an expression for x_+.

(d) Find the earliest time at which the solution blows up.