Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH41920-WE01

Title:
Geometry V

Time:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 20%, Section B is worth 60%, and Section C is worth 20%. Within Sections A and B, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 (a) Let $A_1 = (1:1), A_2 = (1:2) \in \mathbb{RP}^1, B_1 = (1:1:0), B_2 = (1:1:2) \in \mathbb{RP}^2$. Prove or disprove that there exists a projective map $f : \mathbb{RP}^1 \to \mathbb{RP}^2$ with $f(A_k) = B_k$, $k = 1, 2$.

(b) Let $b = \{(x_1 : x_2 : x_3) \in \mathbb{RP}^2 \mid x_1 + x_3 = 0\}$ define a projective line in projective space. Prove or disprove that there exists a bijective map $g : \mathbb{R}^2 \to \mathbb{RP}^2 - b$ with the following property: for any affine map $h : \mathbb{R}^2 \to \mathbb{R}^2$ there exists a projective map $H : \mathbb{RP}^2 \to \mathbb{RP}^2$ satisfying $H \circ g = g \circ h$.

Q2 Let C_1, C_2, C_3 be three concentric circles of radii 1, 2 and 3 in the Euclidean plane. Is there a Möbius transformation mapping C_1, C_2, C_3 to three circles C_1', C_2', C_3', where

(a) C_1', C_2', C_3' are unit circles centred at 0, 4 and $2 + 2i\sqrt{3}$?

(b) C_1', C_2', C_3' are three concentric circles of radii 1,2 and 4?

SECTION B

Q3 (a) Prove or disprove that for any distinct numbers $a, b, c \in \mathbb{R}$ there exists a number $d \in \mathbb{R}$ such that the cross ratio satisfies $[a, b, c, d] = 1$.

(b) Let $A_k, B_k \in \mathbb{RP}^2$ be given by

$$A_1 = (1:2:1), A_2 = (2:1:1), A_3 = (3:3:2), A_4 = (1:-1:0),$$

$$B_1 = (0:0:1), B_2 = (1:0:0), B_3 = (1:0:1), B_4 = (1:1:0).$$

Prove or disprove that there exists a projective transformation of $\mathbb{RP}^2$ which maps B_k to A_k for $k = 1, 2, 3, 4$.

(c) For $A \in \mathbb{RP}^2$ let F_A be the collection of projective transformations of $\mathbb{RP}^2$ which fix the point A. Prove that F_A is a group and that for any $B \in \mathbb{RP}^2$ we have: F_B and F_A are isomorphic.

Q4 (a) Prove or disprove that the group of projective transformations of $\mathbb{RP}^2$ acts transitively on the collection of ordered pairs $\mathbb{RP}^2 \times \mathbb{RP}^2$.

(b) Prove or disprove that the group of projective transformations of $\mathbb{RP}^2$ acts transitively on the collection of lines $\{L_B \mid B = (b_1 : b_2 : b_3) \in \mathbb{RP}^2\}$, where $L_B = \{x \in \mathbb{RP}^2 \mid b_1x_1 + b_2x_2 + b_3x_3 = 0\}$.

(c) State the dual version of the problem of part (b).

Q5 Let γ_1 and γ_2 be two circles in $\mathbb{E}^2$ tangent at the point O. Let C_1 and C_2 be two other circles through O, both orthogonal to γ_1 and γ_2. Let A, B, C, D be the intersection points of γ_i with C_j, $i, j \in \{1, 2\}$ other than the point O.

(a) Show that the points A, B, C, D lie on one line or circle.

(b) Is it always possible to find a circle or line C tangent to each of the four circles $\gamma_1, \gamma_2, C_1, C_2$?

(c) Let $M = \gamma_1 \cup \gamma_2 \cup C_1 \cup C_2$ be the union of the four circles considered above. How many different Möbius transformations are there taking the set M to itself (not necessarily pointwise)? Does the answer depend on the configuration?

Q6 Let $ABCDEF$ be a regular hyperbolic hexagon with sides of length $a > 0$.

(a) Let b be the length of the diagonal AC. Find $\cosh b$.

(b) Let r_1 and r_2 be reflections with respect to the lines AC and AE. Let G be the group generated by r_1 and r_2. Assume $\cosh a = 1 + \frac{\sqrt{2}}{3}$. Is it true that the group G acts discretely on $\mathbb{H}^2$? Justify your answer.

(c) Let s_1, s_2, s_3 be reflections with respect to the lines AC, AD, CF respectively. Find the type of the isometry $f = s_3 s_2 s_1$.

SECTION C

Q7 Let $f(x) = A_f x + b_f$ be an affine transformation of $\mathbb{R}^2$, with $A_f \in GL(2, \mathbb{R}), b_f \in \mathbb{R}^2$.

(a) Assume that A_f has real eigenvalues λ_1, λ_2 satisfying $\lambda_1 \neq \lambda_2$, $\lambda_1 \neq 1$, $\lambda_2 \neq 1$. Prove that f is conjugate via an affine transformation to a diagonal linear map $g(x) = B_f x$, where $B_f \in GL(2, \mathbb{R})$ is a diagonal matrix.

(b) Assume that A_f has a real eigenvalue. Let G be the group generated by f. Prove or disprove that if G is finite then the eigenvalues of A_f have absolute value one.

(c) Assume that 1 is not an eigenvalue of $A_f \in GL(2, \mathbb{R})$. Prove that $f : \mathbb{R}^2 \to \mathbb{R}^2$ has a fixed point.