

## **EXAMINATION PAPER**

Examination Session: May/June

2022

Year:

Exam Code:

MATH41920-WE01

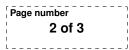
Title:

Geometry V

Time:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 20%, Section B is worth 60%, and Section C is worth 20%. Within Sections A and B, all questions carry equal marks.
	Students must use the mathematics specific answer book.

Revision:



## SECTION A

- **Q1** (a) Let  $A_1 = (1:1), A_2 = (1:2) \in \mathbb{RP}^1, B_1 = (1:1:0), B_2 = (1:1:2) \in \mathbb{RP}^2$ . Prove or disprove that there exists a projective map  $f : \mathbb{RP}^1 \to \mathbb{RP}^2$  with  $f(A_k) = B_k, k = 1, 2$ .
  - (b) Let  $b = \{(x_1 : x_2 : x_3) \in \mathbb{RP}^2 \mid x_1 + x_3 = 0\}$  define a projective line in projective space. Prove or disprove that there exists a bijective map  $g : \mathbb{R}^2 \to \mathbb{RP}^2 b$  with the following property: for any affine map  $h : \mathbb{R}^2 \to \mathbb{R}^2$  there exists a projective map  $H : \mathbb{RP}^2 \to \mathbb{RP}^2$  satisfying  $H \circ g = g \circ h$ .
- Q2 Let  $C_1, C_2, C_3$  be three concentric circles of radii 1, 2 and 3 in the Euclidean plane. Is there a Möbius transformation mapping  $C_1, C_2, C_3$  to three circles  $C'_1, C'_2, C'_3$ , where
  - (a)  $C'_1, C'_2, C'_3$  are unit circles centred at 0, 4 and  $2 + 2i\sqrt{3}$ ?
  - (b)  $C'_1, C'_2, C'_3$  are three concentric circles of radii 1,2 and 4?

## SECTION B

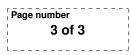
- **Q3** (a) Prove or disprove that for any distinct numbers  $a, b, c \in \mathbb{R}$  there exists a number  $d \in \mathbb{R}$  such that the cross ratio satisfies [a, b, c, d] = 1.
  - (b) Let  $A_k, B_k \in \mathbb{RP}^2$  be given by

$$A_1 = (1:2:1), A_2 = (2:1:1), A_3 = (3:3:2), A_4 = (1:-1:0),$$

 $B_1 = (0:0:1), B_2 = (1:0:0), B_3 = (1:0:1), B_4 = (1:1:0).$ 

Prove or disprove that there exists a projective transformation of  $\mathbb{RP}^2$  which maps  $B_k$  to  $A_k$  for k = 1, 2, 3, 4.

- (c) For  $A \in \mathbb{RP}^2$  let  $F_A$  be the collection of projective transformations of  $\mathbb{RP}^2$ which fix the point A. Prove that  $F_A$  is a group and that for any  $B \in \mathbb{RP}^2$  we have:  $F_B$  and  $F_A$  are isomorphic.
- Q4 (a) Prove or disprove that the group of projective transformations of  $\mathbb{RP}^2$  acts transitively on the collection of ordered pairs  $\mathbb{RP}^2 \times \mathbb{RP}^2$ .
  - (b) Prove or disprove that the group of projective transformations of  $\mathbb{RP}^2$  acts transitively on the collection of lines  $\{L_B \mid B = (b_1 : b_2 : b_3) \in \mathbb{RP}^2\}$ , where  $L_B = \{x \in \mathbb{RP}^2 \mid b_1x_1 + b_2x_2 + b_3x_3 = 0\}.$
  - (c) State the dual version of the problem of part (b).



- **Q5** Let  $\gamma_1$  and  $\gamma_2$  be two circles in  $\mathbb{E}^2$  tangent at the point O. Let  $C_1$  and  $C_2$  be two other circles through O, both orthogonal to  $\gamma_1$  and  $\gamma_2$ . Let A, B, C, D be the intersection points of  $\gamma_i$  with  $C_j$ ,  $i, j \in \{1, 2\}$  other than the point O.
  - (a) Show that the points A, B, C, D lie on one line or circle.
  - (b) Is it always possible to find a circle or line C tangent to each of the four circles  $\gamma_1, \gamma_2, C_1, C_2$ ?
  - (c) Let  $M = \gamma_1 \cup \gamma_2 \cup C_1 \cup C_2$  be the union of the four circles considered above. How many different Möbius transformations are there taking the set M to itself (not necessarily pointwise)? Does the answer depend on the configuration?

**Q6** Let *ABCDEF* be a regular hyperbolic hexagon with sides of length a > 0.

- (a) Let b be the length of the diagonal AC. Find  $\cosh b$ .
- (b) Let  $r_1$  and  $r_2$  be reflections with respect to the lines AC and AE. Let G be the group generated by  $r_1$  and  $r_2$ . Assume  $\cosh a = 1 + \frac{\sqrt{2}}{3}$ . Is it true that the group G acts discretely on  $\mathbb{H}^2$ ? Justify your answer.
- (c) Let  $s_1$ ,  $s_2$ ,  $s_3$  be reflections with respect to the lines AC, AD, CF respectively. Find the type of the isometry  $f = s_3 s_2 s_1$ .

## SECTION C

- **Q7** Let  $f(x) = A_f x + b_f$  be an affine transformation of  $\mathbb{R}^2$ , with  $A_f \in GL(2, \mathbb{R}), b_f \in \mathbb{R}^2$ .
  - (a) Assume that  $A_f$  has real eigenvalues  $\lambda_1$ ,  $\lambda_2$  satisfying  $\lambda_1 \neq \lambda_2$ ,  $\lambda_1 \neq 1$ ,  $\lambda_2 \neq 1$ . Prove that f is conjugate via an affine transformation to a diagonal linear map  $g(x) = B_f x$ , where  $B_f \in GL(2, \mathbb{R})$  is a diagonal matrix.
  - (b) Assume that  $A_f$  has a real eigenvalue. Let G be the group generated by f. Prove or disprove that if G is finite then the eigenvalues of  $A_f$  have absolute value one.
  - (c) Assume that 1 is not an eigenvalue of  $A_f \in GL(2, \mathbb{R})$ . Prove that  $f : \mathbb{R}^2 \to \mathbb{R}^2$  has a fixed point.