Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH4231-WE01

Title:
Statistical Mechanics IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 Consider a thermodynamic system with pressure p, volume V, and temperature T, and with the following internal energy E and equation of state:

$$E(p,T) = \alpha T - \beta p^2 \qquad pV = \gamma T$$

Here α, β and γ are constants, and you can take the first law of thermodynamics to be $dE = TdS - pdV$.

(a) Consider moving the system from (p_0, V_0) to (p_0, V_1), i.e. changing the volume from V_0 to V_1 while holding the pressure fixed at p_0. Compute the change of energy E and of entropy S.

(b) Find a differential equation of the form $\frac{dp}{dV} = f(p,V)$ satisfied by an adiabat in the (p,V) plane. (You do not need to solve this differential equation, but you should work out the form of $f(p,V)$).

(c) What is the change in entropy when moving from (p_0, V_0) to (p_1, V_1), where these two points are connected by an adiabat?

Q2 Consider a particle moving in two dimensions (q_1, q_2) with the following Hamiltonian

$$H(q_i, p_i) = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{m\omega^2}{2}(q_1^2 + q_2^2)$$

Recall that the definition of the Poisson bracket is $\{A, B\} = \sum_{i=1}^{2} \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right)$.

(a) Calculate the following Poisson brackets: $\{q_i, p_j\}$, $\{H, q_i\}$, $\{H, p_j\}$.

(b) Consider the following quantity L:

$$L(q_i, p_i) \equiv q_1 p_2 - q_2 p_1$$

Calculate the Poisson bracket $\{L, H\}$. What does your answer tell you about the time-dependence of L?

(c) In this part of the problem we will study a probability distribution $\rho(q_i, p_i; t)$ which evolves with time under Hamiltonian time evolution. At $t = 0$ this distribution satisfies the following:

$$\rho(q_i, p_i; t = 0) = \frac{1}{\mathcal{N}} \exp(-\alpha L(q_i, p_i))$$

where α and $\mathcal{N}$ are constants and L was defined in the previous part. Use Liouville's theorem to compute $\frac{\partial \rho}{\partial t}$ at $t = 0$. Further use your result to work out $\rho(q_i, p_i; t)$ for all $t > 0$.

Q3 Consider a rigid rotator with Hamiltonian,

$$H = \frac{p_\theta^2}{2\,I} + \frac{p_\phi^2}{2\,I\,\sin^2\theta}\,,$$

in spherical coordinates.

(a) Compute the single particle partition function.

(b) For a system of non-interacting indistinguishable rigid rotators, compute the grand canonical partition function.

Q4 A Quantum Mechanical system has two energy levels E_0 and E_1 with corresponding degeneracies g_0 and g_1. If p_0 and p_1 are the probabilities to find the system in the energy level E_0 and E_1 respectively, compute

(a) The average energy of the system.

(b) The Gibbs entropy.

(c) Specify the probability distribution (that is, the values of p_0 and p_1) which extremise the Gibbs entropy.

SECTION B

Q5 Consider a particle moving in one dimension undergoing a random walk. At each step in the random walk, the particle takes a step where the displacement s is drawn from a probability distribution with density function $w(s)$:

$$w(s) = \begin{cases} \frac{1}{2a}, & s \in [-a, a] \\ 0, & \text{otherwise} \end{cases}$$

(a) Calculate the mean $\langle s \rangle$ and variance $\langle s^2 \rangle_c$ for a single step.

(b) Calculate the characteristic function $\tilde{w}(k)$ associated with the probability distribution for a single step.

(c) After N steps, calculate the mean $\langle X \rangle$ and variance $\langle X^2 \rangle_c$ of the total displacement from the origin X.

(d) Calculate the probability density function for the total displacement X after N steps, $p(X)$. You may leave your answer in the form of an integral.

(e) Write down a closed form expression for $p(X)$ in the limit $N \to \infty$.

Q6 Consider a particle moving in one dimension in a box of size L. The Hamiltonian for the particle is

$$H(q, p) = \frac{p^2}{2m} \qquad q \in [0, L]$$

(a) Write down Hamilton's equations of motion as applied to this particle.

(b) Compute in the microcanonical ensemble the accessible area of phase space $\mathcal{N}(E)$, defined by

$$\mathcal{N}(E) = \int dp dq \delta(H(q, p) - E)$$

(c) Compute the entropy $S(E) = k_B \log \Omega(E)$, where $\Omega(E) = \frac{\mathcal{N}(E)}{\mathcal{N}_0}$ and as usual we take $\mathcal{N}_0 = \frac{h}{E}$ with h a constant. From this, calculate the temperature and the pressure as a function of E. (Note: as the particle moves only in one dimension, the pressure should be viewed as the thermodynamic variable conjugate to the length L).

(d) Calculate $\langle q^2 \rangle_c$ and $\langle p^2 \rangle_c$ in the microcanonical ensemble.

Q7 A system of indistinguishable classical particles of mass m moves inside a one dimensional potential,

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2\,(x+L)^2 & , x < -L \\ 0 & , -L < x < L \\ \frac{1}{2}m\omega^2\,(x-L)^2 & , L < x \end{cases} .$$

(a) Consider the number of particles N and temperature T fixed and compute the partition function.

(b) Compute the average energy for the system in part (a).

(c) Suppose that the number of particles can change and that the system is kept at fixed chemical potential μ. Compute the grand canonical potential Φ and the average number of particles $\langle N \rangle$.

Q8 Consider a free quantum particle which in the continuum limit has energy density of states $g(E)$.

(a) Write the grand canonical potential for a system of non-interacting particles which is held at temperature T and chemical potential μ in terms of $g(E)$, for both the case where all particles are identical Fermions and where all particles are identical Bosons.

(b) Suppose that the particles move in three dimensions with their energy E and momentum $\mathbf{p}$ satisfying the dispersion relation $E = \gamma\,|\boldsymbol{p}|^s$. Given that the numbers γ and s are both positive, show that the equation of state is,

$$PV = \lambda\,\langle E \rangle\,,$$

with P the pressure, $\langle E \rangle$ the average energy and λ a constant you should specify. The above should be shown for both cases of identical Fermions and Bosons.

(c) For the case of Fermions at low temperatures, show that,

$$P \propto \left(\frac{N}{V}\right)^y ,$$

with y a real number you should specify.