

EXAMINATION PAPER

Examination Session: May/June

2022

Year:

Exam Code:

MATH4231-WE01

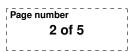
Title:

Statistical Mechanics IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.		
	Students must use the mathematics specific answer book.		

Revision:



SECTION A

Q1 Consider a thermodynamic system with pressure p, volume V, and temperature T, and with the following internal energy E and equation of state:

$$E(p,T) = \alpha T - \beta p^2$$
 $pV = \gamma T$

Here α, β and γ are constants, and you can take the first law of thermodynamics to be dE = TdS - pdV.

- (a) Consider moving the system from (p_0, V_0) to (p_0, V_1) , i.e. changing the volume from V_0 to V_1 while holding the pressure fixed at p_0 . Compute the change of energy E and of entropy S.
- (b) Find a differential equation of the form $\frac{dp}{dV} = f(p, V)$ satisfied by an adiabat in the (p, V) plane. (You do not need to solve this differential equation, but you should work out the form of f(p, V)).
- (c) What is the change in entropy when moving from (p_0, V_0) to (p_1, V_1) , where these two points are connected by an adiabat?
- **Q2** Consider a particle moving in two dimensions (q_1, q_2) with the following Hamiltonian

$$H(q_i, p_i) = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{m\omega^2}{2}(q_1^2 + q_2^2)$$

Recall that the definition of the Poisson bracket is $\{A, B\} = \sum_{i=1}^{2} \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right).$

- (a) Calculate the following Poisson brackets: $\{q_i, p_j\}, \{H, q_i\}, \{H, p_j\}$.
- (b) Consider the following quantity L:

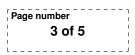
$$L(q_i, p_i) \equiv q_1 p_2 - q_2 p_1$$

Calculate the Poisson bracket $\{L, H\}$. What does your answer tell you about the time-dependence of L?

(c) In this part of the problem we will study a probability distribution $\rho(q_i, p_i; t)$ which evolves with time under Hamiltonian time evolution. At t = 0 this distribution satisfies the following:

$$\rho(q_i, p_i; t = 0) = \frac{1}{\mathcal{N}} \exp(-\alpha L(q_i, p_i))$$

where α and \mathcal{N} are constants and L was defined in the previous part. Use Liouville's theorem to compute $\frac{\partial \rho}{\partial t}$ at t = 0. Further use your result to work out $\rho(q_i, p_i; t)$ for all t > 0.





Q3 Consider a rigid rotator with Hamiltonian,

$$H = \frac{p_{\theta}^2}{2I} + \frac{p_{\phi}^2}{2I\sin^2\theta},$$

in spherical coordinates.

- (a) Compute the single particle partition function.
- (b) For a system of non-interacting indistinguishable rigid rotators, compute the grand canonical partition function.
- **Q4** A Quantum Mechanical system has two energy levels E_0 and E_1 with corresponding degeneracies g_0 and g_1 . If p_0 and p_1 are the probabilities to find the system in the energy level E_0 and E_1 respectively, compute
 - (a) The average energy of the system.
 - (b) The Gibbs entropy.
 - (c) Specify the probability distribution (that is, the values of p_0 and p_1) which extremise the Gibbs entropy.



SECTION B

Q5 Consider a particle moving in one dimension undergoing a random walk. At each step in the random walk, the particle takes a step where the displacement s is drawn from a probability distribution with density function w(s):

$$w(s) = \begin{cases} \frac{1}{2a}, & s \in [-a, a] \\ 0, & \text{otherwise} \end{cases}$$

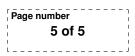
- (a) Calculate the mean $\langle s \rangle$ and variance $\langle s^2 \rangle_c$ for a single step.
- (b) Calculate the characteristic function $\tilde{w}(k)$ associated with the probability distribution for a single step.
- (c) After N steps, calculate the mean $\langle X \rangle$ and variance $\langle X^2 \rangle_c$ of the total displacement from the origin X.
- (d) Calculate the probability density function for the total displacement X after N steps, p(X). You may leave your answer in the form of an integral.
- (e) Write down a closed form expression for p(X) in the limit $N \to \infty$.
- **Q6** Consider a particle moving in one dimension in a box of size L. The Hamiltonian for the particle is

$$H(q,p) = \frac{p^2}{2m} \qquad q \in [0,L]$$

- (a) Write down Hamilton's equations of motion as applied to this particle.
- (b) Compute in the microcanonical ensemble the accessible area of phase space $\mathcal{N}(E)$, defined by

$$\mathcal{N}(E) = \int dp dq \delta(H(q, p) - E)$$

- (c) Compute the entropy $S(E) = k_B \log \Omega(E)$, where $\Omega(E) = \frac{\mathcal{N}(E)}{\mathcal{N}_0}$ and as usual we take $\mathcal{N}_0 = \frac{h}{E}$ with h a constant. From this, calculate the temperature and the pressure as a function of E. (Note: as the particle moves only in one dimension, the pressure should be viewed as the thermodynamic variable conjugate to the length L).
- (d) Calculate $\langle q^2 \rangle_c$ and $\langle p^2 \rangle_c$ in the microcanonical ensemble.





Q7 A system of indistinguishable classical particles of mass m moves inside a one dimensional potential,

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2 (x+L)^2 &, x < -L \\ 0 &, -L < x < L \\ \frac{1}{2}m\omega^2 (x-L)^2 &, L < x \end{cases}$$

- (a) Consider the number of particles N and temperature T fixed and compute the partition function.
- (b) Compute the average energy for the system in part (a).
- (c) Suppose that the number of particles can change and that the system is kept at fixed chemical potential μ . Compute the grand canonical potential Φ and the average number of particles $\langle N \rangle$.
- **Q8** Consider a free quantum particle which in the continuum limit has energy density of states g(E).
 - (a) Write the grand canonical potential for a system of non-interacting particles which is held at temperature T and chemical potential μ in terms of g(E), for both the case where all particles are identical Fermions and where all particles are identical Bosons.
 - (b) Suppose that the particles move in three dimensions with their energy E and momentum **p** satisfying the dispersion relation $E = \gamma |\mathbf{p}|^s$. Given that the numbers γ and s are both positive, show that the equation of state is,

$$PV = \lambda \langle E \rangle$$
,

with P the pressure, $\langle E \rangle$ the average energy and λ a constant you should specify. The above should be shown for both cases of identical Fermions and Bosons.

(c) For the case of Fermions at low temperatures, show that,

$$P \propto \left(\frac{N}{V}\right)^y,$$

with y a real number you should specify.