Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH4241-WE01

Title: Representation Theory IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 Decide whether each of the following statements is true for every character χ of every finite group G, in each case providing a proof or a counterexample.

(a) For all $g \in G$, $|\chi(g)| \leq \chi(1)$.

(b) We have

$$\sum_{g \in G} \chi(g)^2 = |G|.$$

Q2 The following is part of the character table of a group G whose order is 72 and which has six conjugacy classes. The conjugacy classes are labeled by the order of their elements; for example, elements of C_{4A} have order 4.

	C_1	C_2	C_3	C_{4A}	C_{4B}	C_{4C}
χ_0	1	1	1	1	1	1
χ_1	1	1	1	-1	-1	1
χ_2	1	1	1	-1	1	-1
χ_3	1	1	1	1	-1	-1
ψ	2		2			
θ						

(a) Find the missing entries in the character table.

(b) Is G a simple group? Justify your answer.

Q3 Recall that $\mathrm{SL}_2(\mathbb{C})$ has Lie algebra $\mathfrak{sl}_{2,\mathbb{C}}$. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{sl}_{2,\mathbb{C}}$.

(a) Find $\exp(tA)$ for $t \in \mathbb{R}$, giving your answer in terms of trigonometric functions.

(b) Let $V = \mathbb{C}[x, y]$ be the vector space of polynomials in two variables. Let ρ be the representation of $\mathrm{SL}_2(\mathbb{C})$ on V defined by

$$(\rho(g)f)(x, y) = f\left((x, y)g\right)$$

for $g \in \mathrm{SL}_2(\mathbb{C})$ and $f \in V$, where we regard (x, y) as a row vector and multiply it on the right by the matrix g.

Let $D\rho$ be the derived representation of $\mathfrak{sl}_{2,\mathbb{C}}$ on V. Show that

$$D\rho(A) = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Q4 Let $\mathfrak{g} = \mathfrak{sl}_{3,\mathbb{C}}$ and let V be a representation of $\mathfrak{g}$.

(a) State the definitions of a *weight vector* and an *highest weight vector* in V.

(b) Draw the weight diagram for $V = \mathrm{Sym}^2(\mathbb{C}^3)$, where $\mathbb{C}^3$ is the standard representation of $\mathfrak{g}$, showing your working.

Write down a highest weight vector in V, and indicate its weight on your diagram.

SECTION B

Q5 Suppose that G is a finite group and that

$$\rho : G \to \mathrm{GL}_n(\mathbb{R})$$

is a representation on $V = \mathbb{R}^n$. Let $v \cdot w$ denote the usual dot product of two vectors in V.

(a) Consider the linear map

$$\phi : \mathrm{Sym}^2(V) \to \mathbb{R}$$

defined by

$$vw \mapsto \sum_{g \in G} (\rho(g)v) \cdot (\rho(g)w)$$

for all $v, w \in V$. Show that ϕ is a G-homomorphism, where $\mathbb{R}$ is given the trivial action of G.

(b) Show that $\phi(v^2) \neq 0$ for all $v \in V$ with $v \neq 0$.

The quaternion group Q_8 is generated by two elements i and j subject to the relations $i^2 = j^2$, $i^4 = e$, and $ij = ji^{-1}$. Its character table is shown below:

	$\{e\}$	$\{i, i^{-1}\}$	$\{j, j^{-1}\}$	$\{ij, (ij)^{-1}\}$	$\{i^2\}$
χ_0	1	1	1	1	1
χ_1	1	-1	-1	1	1
χ_2	1	-1	1	-1	1
χ_3	1	1	-1	-1	1
ψ	2	0	0	0	-2

(c) Decompose $\mathrm{Sym}^2(\psi)$ and $\Lambda^2(\psi)$ into irreducibles.

(d) Show that there is no representation $\rho : Q_8 \to \mathrm{GL}_2(\mathbb{R})$ with character ψ.

Q6 (a) State Frobenius reciprocity.

(b) If G is a finite group, $H \subset G$ is a subgroup, χ is a character of H, $\mathcal{C}$ is a conjugacy class of G, and $\mathcal{C} \cap H = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_r$ for distinct conjugacy classes $\mathcal{D}_1, \ldots, \mathcal{D}_r$ of H, use part (a) to derive the formula

$$\mathrm{Ind}_H^G(\chi)(\mathcal{C}) = \frac{|G|}{|H|} \sum_{i=1}^{r} \frac{|\mathcal{D}_i|}{|\mathcal{C}|} \chi(\mathcal{D}_i).$$

(c) The following shows a row of the character table of A_5, corresponding to a character χ:

g	e	$(12)(34)$	(123)	(12345)	(12354)
$\chi(g)$	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$

Find the induced character $\mathrm{Ind}_{A_5}^{S_5} \chi$. Is it irreducible?

Q7 Let X, Y, H be the standard basis of $\mathfrak{g} = \mathfrak{sl}_{2,\mathbb{C}}$ and let (ρ, V) be an irreducible $\mathbb{C}$-linear representation of $\mathfrak{g}$.

(a) Prove that, if $v \in V$ is a weight vector of weight λ, then $\rho(Y)v$ is a weight vector (if it is nonzero), and find its weight.

(b) Suppose that $v \in V$ is a weight vector of weight λ and that $\rho(X)v = 0$.
Show that

$$\rho(X)\rho(Y)^n v = n(\lambda - n + 1)\rho(Y)^{n-1}v$$

for all integers $n \geq 1$.
Deduce that, if V is finite dimensional, then $\lambda \in \mathbb{Z}$.

(c) Suppose that $V = \mathfrak{g}$ with the adjoint representation. Decompose $\mathrm{Sym}^2(V)$ into irreducibles, and find a vector $v \in \mathrm{Sym}^2(V)$ such that $Av = 0$ for all $A \in \mathfrak{g}$

Q8 Let $J = \begin{pmatrix} \mathbf{0}_2 & I_2 \\ -I_2 & \mathbf{0}_2 \end{pmatrix}$ where I_2 is the 2×2 identity matrix and $\mathbf{0}_2$ is the 2×2 zero matrix. Define a Lie group G by

$$G = \{g \in \mathrm{GL}_4(\mathbb{R}) : g^T J g = J\}.$$

(a) Show that the Lie algebra of G is

$$\mathfrak{g} = \{X \in \mathfrak{gl}_{4,\mathbb{R}} : X^T J + JX = 0\}.$$

(b) Show that a general element of $\mathfrak{g}$ may be written as

$$\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$$

where $A, B, C \in \mathfrak{gl}_{2,\mathbb{R}}$, $B = B^T$, and $C = C^T$.
Hence, or otherwise, find $\dim \mathfrak{g}$.

(c) Let $\mathfrak{h} \subset \mathfrak{g}$ be the subalgebra consisting of diagonal matrices of the form

$$H = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -y \end{pmatrix}$$

for $x, y \in \mathbb{R}$. Find elements E_1, E_2, E_3 of $\mathfrak{g}$ such that, for H of this form, $\mathrm{ad}_H(E_1) = (x+y)E_1$, $\mathrm{ad}_H(E_2) = (x-y)E_2$, and $\mathrm{ad}_H(E_3) = 2xE_2$.
Write down the remaining eigenvalues of ad_H on $\mathfrak{g}$. You do not need to justify your answer.