EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH4361-WE01

Title:
Ergodic Theory and Dynamics IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 In all of this problem, we assume that (X, T) is a topological dynamical system where T is surjective.

(a) When do we call (X, T) *transitive* and when do we call (X, T) *minimal*? When do we call a function $f : X \to \mathbb{R}$ *invariant*?

(b) Show that if (X, T) is transitive, then every continuous invariant function is constant. Is transitivity necessary in this context? Justify your answer with a proof or a counter-example.

(c) Show that (X, T) is minimal if and only if every upper semicontinuous T-invariant function is constant.

Note: A function $f : X \to \mathbb{R}$ is called upper semicontinuous *if for each $x \in X$, we have* $\limsup_{y\to x} f(y) \leq f(x)$.

Hint: Observe that if $A \subseteq X$ is an invariant set, then $\mathbf{1}_A$ is upper semicontinuous.

Q2 In this problem, we use our understanding of systems with discrete spectrum to identify conjugate irrational rotations.

(a) Define the notions *topological eigenfunction*, *topological eigenvalue* and *discrete topological spectrum*.

(b) Suppose we are given two topologically transitive dynamical systems (X_1, T_1) and (X_2, T_2) with discrete topological spectrum. State a necessary and sufficient condition for these systems to be conjugate.

(c) Let $(\mathbb{T}^1, \alpha)$ and $(\mathbb{T}^1, \omega)$ be irrational rotations. When are $(\mathbb{T}^1, \alpha)$ and $(\mathbb{T}^1, \omega)$ conjugate? Justify your answer and in each case, give some conjugacy between $(\mathbb{T}^1, \alpha)$ and $(\mathbb{T}^1, \omega)$.

Q3 Let Σ_A^+ be a one-sided subshift of finite type, $\sigma : \Sigma_A^+ \to \Sigma_A^+$ the shift map and A a primitive matrix. Equip Σ_A^+ with the metric

$$d(x, y) = 2^{-\min\{i\in\mathbb{N} : x_i \neq y_i\}}.$$

(a) Show that $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is positively expansive.

(b) Let P be a stochastic matrix assumed to be compatible with A.

(i) State the definition of the Markov measure μ_P associated to P.

(ii) Show that μ_P is shift invariant.

Q4 (a) Let (X, d) be a compact metric space and T a continuous map. State the definition of the topological entropy $h_{\text{top}}(T)$ of T.

(b) Equip $\mathbb{T}^1$ with the metric $d(x, y) = \min\{|x - y + m| : m \in \mathbb{Z}\}$. Show that any rotation $R_\alpha : \mathbb{T}^1 \to \mathbb{T}^1$, $x \mapsto \alpha + x \bmod 1$, has $h_{\text{top}}(R_\alpha) = 0$.

(c) Equip $\mathbb{T}^2$ with the metric $\rho(v \bmod 1, w \bmod 1) = \min\{\|\boldsymbol{v} - \boldsymbol{w} + \boldsymbol{m}\|_\infty : \boldsymbol{m} \in \mathbb{Z}^2\}$ with maximum norm given by $\|\boldsymbol{v}\|_\infty = \max(|v_1|, |v_2|)$ for any $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Let

$T_A : \mathbb{T}^2 \to \mathbb{T}^2$ be the linear toral automorphism with matrix $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Show that $h_{\text{top}}(T_A) = 0$.
Hint: show that

$$\rho(T_A^n \boldsymbol{v} \bmod 1, T_A^n \boldsymbol{w} \bmod 1) \leq 2n\rho(\boldsymbol{v} \bmod 1, \boldsymbol{w} \bmod 1).$$

You may assume that the metric balls for ρ are rectangles. One can bound the size of a separated set using the area of the rectangles.

SECTION B

Q5 In this problem, we discuss several (not necessarily related) facts regarding minimal systems.

(a) Let (X, T) be a topological dynamical system. Show that two minimal sets M_1 and M_2 of (X, T) are either disjoint or coincide.
Recall that we call a non-empty set $M \subseteq X$ minimal if $(M, T|_M)$ is minimal.

(b) Let (X, T) be a minimal topological dynamical system. Show that for each $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that for any $x \in X$ we have that

$$\{T^\ell(x) : \ell = 0, 1, \ldots, n\}$$

is ε-dense in X.
Note: We call a subset Y of a metric space (X, d) ε-dense *if for each $x \in X$ there is $y \in Y$ with $d(x, y) \leq \varepsilon$.*

(c) Let $([0, 1], T)$ be a topological dynamical system on the unit interval. Show that $([0, 1], T)$ is not minimal.

Q6 In this problem, we take a look at the relation between topological mixing and equicontinuity.

(a) When do we call a dynamical system *topologically mixing*? Give an example of such a system and briefly describe why it is topologically mixing.

(b) Show that if a system (X, T) is topologically mixing, then so is each of its factors.

(c) When do we call a topological dynamical system *equicontinuous*? Show that non-trivial equicontinuous systems (that is, equicontinuous systems other than the one-point system) are not topologically mixing.

(d) Show that every topological eigenfunction $f : X \to S^1$ of a topologically transitive dynamical system (X, T) can be interpreted as a factor map onto a system defined on S^1 or a subset of S^1. Conclude that topologically mixing systems have no non-constant eigenfunction.

Q7 Let $(X, \mathcal{B}, \mu)$ be a Borel probability space and T a measure preserving transformation

(a) State the definition of the measure theoretic entropy $h_\mu(T)$ of T.

(b) Let ξ and η be countable measurable partitions. State what it means for ξ to be *contained* in γ and show that if ξ is contained in γ then the conditional information $I_\mu(\xi|\sigma(\gamma))$ is equal to 0 μ almost surely.

(c) State what it means for ξ to be a strong generator for T.

(d) Show that if T is invertible and has a strong generator then $h_\mu(T) = 0$.

Q8 Let $X = \{0,1\}^{\mathbb{N}}$. Let m be a Borel probability measure.

(a) Let $\{A_1, \cdots, A_k\}$ be a finite measurable partition of X.

(i) Show that the following inequality holds

$$-\sum_{j=1}^{k} m(A_j) \log m(A_j) \leq \log k.$$

(ii) Show that if the partition $\{A_1, \cdots, A_k\}$ has

$$-\sum_{j=1}^{k} m(A_j) \log m(A_j) = \log k,$$

then we must have $m(A_j) = \frac{1}{k}$ for each $j = 1, \ldots, k$.
Hint: you may use that $\phi(x) = x \log x$ is strictly convex.

(b) Show that $h_m(T) = \log 2$ implies that m equals the $(\frac{1}{2}, \frac{1}{2})$ Bernoulli measure.

(c) Deduce that $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ has a unique measure of maximal entropy.

(d) Show that the doubling map $E_2 : \mathbb{T}^1 \to \mathbb{T}^1$ has a unique measure of maximal entropy.