Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH4371-WE01

Title: Functional Analysis and Applications IV

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 **1.1** State the Hahn-Banach theorem.

1.2 Let $\mathcal{X}$ be a Banach space over a field $\mathbb{F}$. For a given $0 \neq x \in \mathcal{X}$ define

$$\widetilde{f_x} : \operatorname{span}\{x\} \to \mathbb{F}$$

by $\widetilde{f_x}(\alpha x) = \alpha \|x\|$ where $\alpha \in \mathbb{F}$. Show that $\widetilde{f_x}$ is a linear functional and that $\left\|\widetilde{f_x}\right\|_{\operatorname{span}\{x\}^*} = 1$.

1.3 Prove that for any $0 \neq x \in \mathcal{X}$ there exists $f_x \in \mathcal{X}^*$ such that $f_x(x) = \|x\|$ and $\|f_x\|_{\mathcal{X}^*} = 1$.

Q2 Consider the operator $T : \ell_\infty \to \ell_p$, with $1 \leq p < \infty$, defined by

$$T(\boldsymbol{a}) = \left(a_1, \frac{a_2}{2^\alpha}, \dots, \frac{a_n}{n^\alpha}, \dots\right).$$

2.1 Show that T is well defined when $\alpha > 1/p$. In that case also show that it is a linear operator and that it is bounded. Is T well defined when $\alpha = 1/p$?

2.2 Show that for *any* $\alpha > 1/p$ the operator T is injective but not surjective.

Q3 Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces.

3.1 Define what it means for a linear operator $T : \mathcal{X} \to \mathcal{Y}$ to be compact.

3.2 Let $f : \mathcal{X} \to \mathbb{C}$ be an unbounded linear functional (where we assume that such a functional exists). For a fixed $0 \neq x_0 \in \mathcal{X}$ let $T : \mathcal{X} \to \mathcal{X}$ be defined by $Tx = f(x)x_0$. Show that T has finite rank (i.e. $\dim(\mathcal{R}(T)) < \infty$) but is not compact.

Q4 Let $u : \mathbb{R} \to \mathbb{R}$ be defined by $u(x) = (1 + |x|)^{-1}$.

4.1 For which $1 \leq p \leq \infty$ is $u \in W_0^{1,p}(\mathbb{R})$?

4.2 Give an explicit $f \in H^{-1}(\mathbb{R})$ such that $f(u) = 1$.

SECTION B

Q5 Consider the subset $\mathcal{H} \subset \ell_2$ given by

$$\mathcal{H} = \left\{ \boldsymbol{a} \in \ell_2 \mid \sum_{n\in\mathbb{N}} n^2 \left|a_n\right|^2 < \infty \right\}.$$

5.1 Is $\mathcal{H}$ closed with respect to the norm of ℓ_2? Prove your claim.

5.2 Let B be the set

$$B = \left\{ \boldsymbol{a} \in \mathcal{H} \mid \sum_{n\in\mathbb{N}} n^2 \left|a_n\right|^2 \leq 1 \right\} \subset \mathcal{H}.$$

Show that for any $\boldsymbol{a} \in B$ we have that

$$\sum_{n\geq N} \left|a_n\right|^2 \leq \frac{1}{N^2}$$

and then prove that if $\{\boldsymbol{a}_n\}_{n\in\mathbb{N}}$ is a sequence in B such that

$$(a_n)_j \underset{n\to\infty}{\longrightarrow} a_j, \qquad \forall j \in \mathbb{N}$$

for some $\boldsymbol{a} \in B$ (component-wise convergence) then

$$\left\|\boldsymbol{a}_n - \boldsymbol{a}\right\|_{\ell_2} \underset{n\to\infty}{\longrightarrow} 0.$$

Q6 Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces and let E be a given subset of $\mathcal{X}$. Let $T : \mathcal{X} \to \mathcal{Y}$ be a given bounded linear operator.

6.1 Show that if $T|_E = 0$ then $T|_{\mathcal{M}} = 0$ where $\mathcal{M} = \overline{\mathrm{span} E}$.

6.2 Let $\{E_n\}_{n\in\mathbb{N}}$ be a given sequence of subsets of $\mathcal{X}$. Show that if $\{T_n\}_{n\in\mathbb{N}} \in B(\mathcal{X}, \mathcal{Y})$ satisfy

$$T_n|_{E_j} = 0, \qquad \forall n \geq j,$$

then if $\{T_n\}_{n\in\mathbb{N}}$ converges to $T \in B(\mathcal{X}, \mathcal{Y})$ in the operator norm we have that

$$T|_{\overline{\mathrm{span} \cup_{n\in\mathbb{N}} E_n}} = 0.$$

Q7 Consider $T : \ell_2 \to \ell_2$ defined by

$$T(x_1, x_2, x_3, x_4, x_5, x_6, \dots) = (0, x_1, 0, x_3, 0, x_5, \dots).$$

7.1 Compute T^2.

7.2 Find $\rho(T)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$.

Q8 Let $T = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$ in the Hilbert space $L^2([0,1])$ with

$$\mathcal{D}(T) = \left\{ f \in L^2([0,1]) : f \text{ absolutely continuous on } [0,1],\ f' \in L^2([0,1]),\ f(0) = \mathrm{i}f(1) \right\}.$$

8.1 Show that T is densely defined.

8.2 Show that T is symmetric.

8.3 Show that T is selfadjoint. You may use without proof that the general solution of $f' - \mu f = g$ (for given $\mu \in \mathbb{C}$ and $g \in L^2([0,1])$) is

$$f(x) = \exp(\mu x) \left(C + \int_0^x \exp(-\mu t) g(t)\, \mathrm{d}t \right),$$

for a constant $C \in \mathbb{C}$.