Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2022	MATH4381-WE01

Title:

Topics in Applied Mathematics IV

Time:	3 hours	
Additional Material provided:	Formula sheet.	
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2022

SECTION A

Q1 A magnetic field is given by $\boldsymbol{B} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + (3y - 2z)\boldsymbol{e}_z$.

(a) Find the corresponding current density and Lorentz force.

(b) Could this be (i) a force-free equilibrium, or (ii) a magnetostatic equilibrium?

Q2 A two-dimensional magnetic field has the form $\boldsymbol{B}(x, y, t) = \nabla A(x, y, t) \times \boldsymbol{e}_z$, and evolves under the ideal induction equation with a velocity field of the form $\boldsymbol{u} = \nabla\psi(x, y, t) \times \boldsymbol{e}_z$.

(a) Write down the ideal induction equation for $\boldsymbol{B}$ and show that

$$\frac{\partial A}{\partial t} = -\boldsymbol{u} \cdot \nabla A + a_0$$

for some constant a_0.

(b) If $S \subset \mathbb{R}^2$ is a fixed region with $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on the boundary (where $\boldsymbol{n}$ is the normal direction on the boundary), show that $\int_S A^2 \, \mathrm{d}S$ is an ideal invariant.

Q3 Consider an expanding and incompressible elliptical shell defined by the following map:

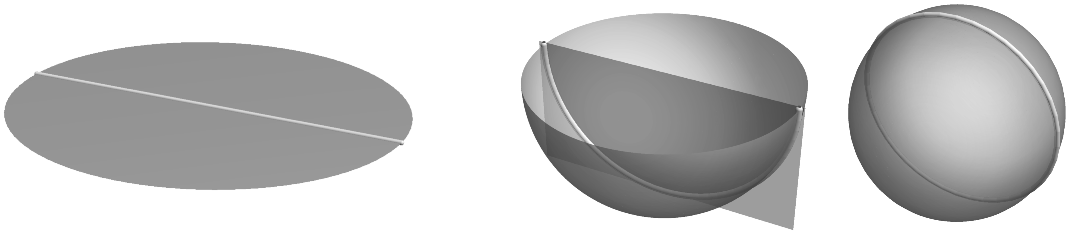
$$\begin{aligned}
x_1(R, \theta, \phi) &= r_1(R)\sin(\theta)\cos(\phi),\\
x_2(R, \theta, \phi) &= r_1(R)\sin(\theta)\sin(\phi),\\
x_3(R, \theta, \phi) &= r_2(R)\cos(\theta).
\end{aligned}$$

Here $R \in [A, B]$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$, and $r_1(R) = r_2(R) = R$ in the undeformed state. Assume further that the shell is an **isotropic** hyperelastic material whose Cauchy stress tensor takes the form

$$\Sigma = \mathrm{F}\frac{\partial W}{\partial \mathrm{F}} - p\mathrm{I}.$$

Show that the critical stress differences $\sigma_{11} - \sigma_{33}$ and $\sigma_{22} - \sigma_{33}$ are the same as the spherical expansion case $r_1 = r_2$, if $r_1/R = \sqrt{R}/\sqrt{r_2}$.

You may quote any results derived in class regarding the stress differences in that case.



(a) (b) (c)

Figure 1: The deformation of a disc into a sphere passing continuously through spherical segments (**Q4**): (a) the original disc; (b) a half sphere; (c) the final sphere. The curve shown on all figures is the intersection of the surface and of a plane though the pole of the surface (as indicated in panel (b)). The length of this curve should be fixed during the deformation.

Q4 (a) Describe a continuously parameterised deformation of a disc into a sphere, as depicted in Figure 1, which has the following properties. First, the body always represents a segment of a sphere with a circular boundary of constant height (see Figure 1(b)). Second, the arclength of the curve shown on Figure 1 is preserved. This curve represents the intersection of the surface and a vertical plane though the sphere's pole, as shown in Figure 1(b).

(b) Calculate the Cauchy deformation tensor of the body described in (a). Describe any strains induced in the body during this deformation.

SECTION B

Q5 Let D be the region $x \in [-\pi, \pi]$, $y \in [-\pi, \pi]$.

(a) Define what is meant by a potential magnetic field.

(b) Find the potential magnetic field $\boldsymbol{B}(x, y)$ in D, satisfying the boundary conditions

$$B_x(-\pi, y) = B_x(\pi, y) = 0, \quad B_y(x, -\pi) = \cos x, \quad B_y(x, \pi) = 0,$$

and sketch the magnetic field lines.

(c) Using your result from (b), write down the potential field in D satisfying

$$B_x(-\pi, y) = B_x(\pi, y) = b_0, \quad B_y(x, -\pi) = \cos x, \quad B_y(x, \pi) = 0,$$

where b_0 is a constant. (You need not sketch the field lines in this case.)

Q6 In this question, we are interested in the possibility of dynamo action in a sphere $r < 1$ by a steady, incompressible velocity field $\boldsymbol{u} = u_\theta(r, \theta, \phi)\boldsymbol{e}_\theta + u_\phi(r, \theta, \phi)\boldsymbol{e}_\phi$. Let $q = \boldsymbol{x} \cdot \boldsymbol{B} = rB_r$, and assume the boundary condition $\boldsymbol{B}(1, \theta, \phi, t) = \boldsymbol{0}$.

(a) Use the (resistive) induction equation to show that

$$\frac{\partial q}{\partial t} = -\boldsymbol{u} \cdot \nabla q + \eta \Delta q.$$

(b) If V is the interior of the sphere, show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \frac{q^2}{2} \, \mathrm{d}V = -\eta \int_V |\nabla q|^2 \, \mathrm{d}V,$$

and hence argue that $|B_r| \to 0$ throughout V as $t \to \infty$.

(c) Now write $\boldsymbol{B} = \nabla \times (T\boldsymbol{x})$. Show that

$$\frac{\partial T}{\partial t} = -\boldsymbol{u} \cdot \nabla T + \eta \Delta T + f,$$

for some scalar function $f(r, t)$ that depends on the particular choice of T.

[You may use without proof that $\Delta(T\boldsymbol{x}) = \boldsymbol{x}\Delta T + 2\nabla T$.]

(d) Hence prove that such a dynamo is impossible; *i.e.*, $|\boldsymbol{B}| \to 0$ inside the sphere as $t \to \infty$.

[It may help to choose the particular T with zero mean on every surface of constant r.]

Q7 Consider the following deformation of a hyperelastic body $\mathcal{B}$:

$$x_1(R,\Phi,Z) = r(Z)R\cos(\Phi),$$
$$x_2(R,\Phi,Z) = r(Z)R\sin(\Phi),$$
$$x_3(R,\Phi,Z) = z(Z).$$

The undeformed body is described by the parameter ranges $R \in [0,1]$, $\Phi \in [0,2\pi)$, $Z \in [0,H]$ and in this undeformed state $r(Z)=1$, $z(Z)=Z$.

(a) Show the (cylindrical) Cauchy-Green strain tensor $\mathrm{E} = (\mathrm{C}-\mathrm{C}^0)/2$ of this body is:

$$\mathrm{E} = \frac{1}{2}\begin{pmatrix} r^2-1 & 0 & r'rR \\ 0 & R^2(r^2-1) & 0 \\ r'rR & 0 & R^2(r')^2+(z')^2-1 \end{pmatrix}, \qquad (1)$$

where $r' = \mathrm{d}r/\mathrm{d}Z$ and $z' = \mathrm{d}z/\mathrm{d}Z$.

(b) Describe the strain indicated by the form of E given in (1), with reference to the permissible geometry of the deformed body.

(c) Assuming the body is incompressible so $z' = 1/r^2$, and using an appropriate strain energy function, the Euler-Lagrange equations of this system lead to the following ODE:

$$-\frac{20(k_1+3N)}{r^3} + k_2\left[\left(8-3r^2\right)r'' - 3r\left(r'^2+10\right) + 10r^3\right] + \frac{40k_1}{r^5} = 0. \qquad (2)$$

Here N is an applied stretching load. An initially cylindrical body is observed to deform under the action of a small stretching load $N \ll 1$ to form a necking configuration where it stretches axially and contracts radially, as shown in Figure 2. At the body's caps $z=0$ and $z=H$ there is no radial stretching.

Demonstrate that the solutions to equation (2) indicate it is an appropriate model for this phenomenon.

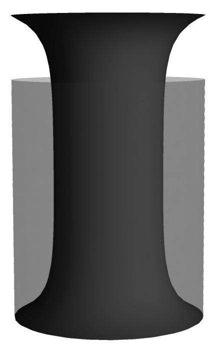


Figure 2: For **Q7**(c). A initially cylindrical body (opaque) deforms by stretching and thinning around its midriff under the action of an applied stretching load.

Q8 Consider a surface $\mathbf{x}(s,h)$ parameterised as

$$\mathbf{x}(s,h) = \mathbf{r}(s) + h\big[\mathbf{b} + \eta(s)\mathbf{t}(s)\big],$$

where $h \in [-H, H]$, $s \in [0, L]$, and s is the arclength parameter for the three-dimensional curve $\mathbf{r}(s)$ which lies along the centre of the surface. The curve has an associated (Frenet) orthonormal triad of unit vectors composed of the unit tangent vector $\mathbf{t} = \mathrm{d}\mathbf{r}/\mathrm{d}s$, and the normal $\mathbf{n}$ and binormal $\mathbf{b}$ vectors, which satisfy the following system of equations:

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \kappa\mathbf{n}, \qquad \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} = -\kappa\mathbf{t} + \eta\kappa\mathbf{b}$$
$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = -\eta\kappa\mathbf{n}, \qquad \kappa = \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s}\cdot\mathbf{n}.$$

(a) Show that the metric (Cauchy deformation) tensor of this body takes the form

$$\mathrm{C} = \begin{pmatrix} (1+t\eta')^2 & \eta(1+t\eta') \\ \eta(1+t\eta') & 1+\eta^2 \end{pmatrix}.$$

(b) Show that the Weingarten tensor (shape tensor) takes the form

$$\mathrm{D} = \begin{pmatrix} -\kappa(1+\eta^2)/(1+h\eta') & -\kappa\eta \\ 0 & 0 \end{pmatrix}.$$

(c) Show this body transforms isometrically for all smooth $\mathbf{r}$ and η.

(d) Under suitable approximations it can be shown that the elastic energy functional of this system, under the action of an applied load N, takes the form

$$E(\theta, \theta'(s)) = \int_0^L \left[N\cos\theta(s) + \frac{1}{\theta'(s)^2}\right]\mathrm{d}s,$$

where θ is the polar angle of the unit tangent vector $\mathbf{t}$. Find the family of functions $\theta(s)$ which are extrema of this energy functional (you may leave solutions in integral form).

Hint: you should use the substitution $v(\theta) = \mathrm{d}\theta/\mathrm{d}s$.