Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH1051-WE01

Title:
Analysis I

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

Q1 (a) Let $a, b \in \mathbb{R}$ with $a \leq b$. Let $a^* = \max\{0, a\}$ and $b^* = \max\{0, b\}$. Show that

$$b^* - a^* \leq b - a.$$

(b) Let $a, b, c \in \mathbb{R}$ with $a < b < c$. Show that

$$\frac{a|b-c| + b|a-c| + c|a-b|}{|a-b| + |a-c| + |b-c|} = b.$$

Q2 Recall that for a function $f \colon \mathbb{R} \to \mathbb{R}$ and $A \subset \mathbb{R}$ the *image of A under f* is

$$f(A) = \{f(a) \in \mathbb{R} \mid a \in A\}.$$

Work out $\sup(f(A))$ and $\inf(f(A))$ in the following cases. Also, decide whether $f(A)$ has a maximum or a minimum in these cases.

(a) Let $f(x) = x^2 - 2x + 2$ and $A = (0, 2)$.

(b) Let $f(x) = 3 - 4x$ and $A = [0, 5) \cup (6, 7)$.

Q3 (a) For the following sequences, calculate the limit or show that no limit exists. State any results that you use.

(i) $x_n = \left(1 - \dfrac{1}{n^3}\right)^{n^2}$.

(ii) $y_n = \dfrac{e^{-n} + \frac{n+1}{2n+1}}{\frac{\log n}{n} - \frac{2n^2-1}{n^2+1}}$.

(b) Let $0 < \alpha < 1$, $a_0 = 0$, $a_1 = 1$, and for $n \geq 2$ let $a_n = \alpha a_{n-1} + (1-\alpha) a_{n-2}$.

(i) Show that $|a_n - a_{n-1}| = (1-\alpha)^{n-1}$ for $n \geq 2$.

(ii) Show that $(a_n)_{n \in \mathbb{N}}$ is a Cauchy sequence.

Q4 (a) Let

$$x_n = \begin{cases} \left(1 + \frac{1}{n^2}\right)^n & n = 3k \\ (-1)^n \sqrt{n-1}(\sqrt{n+1} - \sqrt{n}) & n = 3k-1 \\ e^{-n} & n = 3k-2 \end{cases}$$

Calculate $\limsup\limits_{n \to \infty} x_n$ and $\liminf\limits_{n \to \infty} x_n$.

(b) Let $(a_n)_{n \in \mathbb{N}}$ be a bounded sequence, and $(a_{n_j})_{j \in \mathbb{N}}$ a subsequence of it.

(i) Show that

$$\limsup_{j \to \infty} a_{n_j} \leq \limsup_{n \to \infty} a_n.$$

(ii) Give an example of a bounded sequence $(a_n)_{n \in \mathbb{N}}$ and a subsequence $(a_{n_j})_{j \in \mathbb{N}}$ such that

$$\limsup_{j \to \infty} a_{n_j} \neq \limsup_{n \to \infty} a_n.$$

Q5 Determine whether the following series are convergent or not. State any results that you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(5^n+1)\cdot(n!)^2}{(2n)!}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^{n^2}$.

(c) $\displaystyle\sum_{n=2}^{\infty} \log\left(1+\frac{(-1)^n}{n}\right)$.

Q6 (a) State the ϵ-δ definition of continuity of a function $f : X \to \mathbb{R}$ at a point c.

(b) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by the formula

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

(i) Determine the set of points $c \in \mathbb{R}$ at which $f(x)$ is continuous.

(ii) Determine the set of points at which $f(x)$ is differentiable.

Q7 (a) State Taylor's Theorem with the Lagrange form of the remainder.

(b) Consider the function $f(x) = x^4 + 9x^3 + 7x^2 + 2x - 1$.

(i) Compute the 2nd order Taylor polynomial $T^{(2)}_{f,c}(x)$ for $c = 0$.

(ii) Determine ξ as a function of x in the Lagrange form of the remainder for the above Taylor polynomial.

Q8 For each of the following infinite series, determine the set of all $x \in \mathbb{R}$ for which it converges. Determine where the convergence is absolute and where it is conditional. State explicitly what results from lectures you are using.

(a) $\sum_{k=0}^{\infty} \left(\frac{3^k}{4^k+9}\right) x^k$.

(b) $\sum_{k=0}^{\infty} a_k(x-7)^k$ with $a_k = \begin{cases} k & \text{if } k \text{ is prime;} \\ 0 & k \text{ is not prime.} \end{cases}$

Q9 (a) State the definition for a sequence of functions $f_n(x)$ on a domain X to converge uniformly.

(b) Prove that the series $\sum_{k=0}^{\infty} \frac{\cos(x^k)}{k(k+x^2)}$ defines a continuous function on $\mathbb{R}$.

(c) Give an example, with proof, of a function $f(x)$ and a sequence of functions $f_n(x)$ on $\mathbb{R}$ that converges pointwise to f but not uniformly.

Q10 (a) State the definition of a regulated function on an interval $[a, b]$ and explain how we define the integral of a regulated function. You do not need to give a complete proof that the integral is well-defined, but you should indicate what needs to be checked.

(b) Show that $\exp(x)$ is a regulated function on $[0, 1]$ by writing down an explicit sequence of partitions and step functions and proving that they converge uniformly.