

EXAMINATION PAPER

Examination Session: May/June

2023

Year:

Exam Code:

MATH1061-WE01

Title:

Calculus I

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:

Q1 1.1 Evaluate the following limit or explain why it does not exist.

$$\lim_{x \to 0} \cos\left(e^{1/x^2}\right) \sin x \; .$$

You may use any valid method.

1.2 Evaluate the following limit or explain why it does not exist.

$$\lim_{x \to 0} \frac{x \sin(\sinh x)}{e^{x^2} - 1}$$

You may use any valid method.

1.3 Calculate $f'(\pi/4)$ where

$$f(\theta) = \int_{\sin^2 \theta}^{\cos^2 \theta} e^{-x^2} dx \; .$$

Q2 (a) Find a recursion relation for

$$I_n = \int_{-\pi}^{\pi} x^n \cos x dx \; .$$

- (b) Evaluate I_6 and I_{13} .
- Q3 Use a change of variables

$$x = \frac{1}{2}u\sin v , \ y = \frac{1}{3}u\cos v$$

to evaluate

$$\int \int_D \frac{xy}{4x^2 + 9y^2} dxdy$$

where D is the part of the interior of the curve $4x^2 + 9y^2 = 25$ with x > 0 and y > 0.

Q4 4.1 Find the function g(x) satisfying

$$g'(x) = g(x) + e^{2x}$$
, $g(0) = 3$.

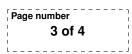
- **4.2** Calculate the Taylor polynomial of degree 2 about $x = \log \pi$ for the function $f(x) = \cos (e^x)$.
- **Q5** (a) Find the Fourier series for the function f with period 4 which is defined by

$$f(x) = \begin{cases} 0 & \text{for } 1 \le |x| \le 2\\ 1 - |x| & \text{for } |x| < 1 \end{cases}$$

(b) Use the Fourier series from part (a) to evaluate

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \; .$$

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Q6 6.1 A function is defined to be

$$f(x,y) = 100 - \frac{1}{15} (2x^2 + 3xy + y^2).$$

Find all unit vectors $\hat{\mathbf{n}}$ such that the directional derivative of f in the direction of $\hat{\mathbf{n}}$ at the point (x, y) = (2, -3) is equal to 1/25.

6.2 Find and classify the stationary points of the function

$$f(x, y, z) = 2x^{3} + 2x^{2} + y^{2} + z^{2} - 2xy - 4xz.$$

Q7 A change of coordinates is given by

$$u(x,y) = \sqrt{x} + \sqrt{y}$$
$$v(x,y) = \sqrt{x} - \sqrt{y},$$

where x and y are both positive.

- (a) Use the Chain Rule to express f_u , f_v and f_{uv} in terms of x and y.
- (b) Using your answer to part a) or otherwise, show that

$$f(x,y) = F(\sqrt{x} + \sqrt{y}) + G(\sqrt{x} - \sqrt{y}),$$

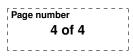
where F and G are arbitrary functions, is a solution to the PDE

$$\frac{f_{xx}}{y} - \frac{f_{yy}}{x} + \frac{f_x - f_y}{2xy} = 0.$$

 $\mathbf{Q8}$ Consider the ordinary differential equation

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - \lambda y = 0.$$

- (a) Explain what is meant by a *regular point* of an ODE. Which points are regular for this ODE?
- (b) A solution to the equation can be expressed in the form $y = \sum_{n=0}^{\infty} a_n x^n$. Find a recurrence relation for the coefficients a_n .
- (c) In the case that $\lambda = 4$, find a formula for the 'even' coefficients a_{2n} in terms of a_0 , and use your answer to find an explicit solution (i.e. not as a series) to the differential equation for this value of λ . Hint: it may help to consider the Taylor series for the exponential function.





- **Q9** A bar of metal lies along the x-axis between x = 0 and $x = \pi$. Its temperature u(x,t) obeys the heat equation $u_t = k^2 u_{xx}$, and its ends are held at zero degrees so that $u(0,t) = u(\pi,t) = 0$.
 - (a) Show that the function

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(nx) e^{-k^2 n^2 t}$$

satisfies the heat equation and the boundary conditions at x = 0 and $x = \pi$.

(b) In the case that the bar initially has temperature

$$u(x,0) = f(x) = \begin{cases} 100 \text{ for } 0 < x \le \pi/2 \\ 0 \text{ for } \pi/2 < x < \pi \end{cases}$$

find the coefficients A_n and hence the solution u(x, t) for t > 0.

- (c) The total heat in the metal bar is given as $H(t) = \int_0^{\pi} u(x,t)dx$. Given the initial condition in part b), find an estimate for the time required for the bar to lose 90% of the initial heat it has at t = 0 in terms of k.
- **Q10** The function y(x) satisfies the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = f(x).$$

- (a) By taking the Fourier transform of this equation with respect to x, find an expression for $\tilde{y}(p)$, the Fourier transform of y(x) in terms of $\tilde{f}(p)$, the Fourier transform of f(x).
- (b) Using your result for part a), find an expression for y(x) in the form of a convolution

$$y(x) = f(x) \star G(x)$$

where you should determine the function G(x) explicitly. It may help to use the fact that the Fourier transform of the one sided exponential

$$h(x) = \begin{cases} e^{-\alpha x} & \text{ for } x > 0\\ 0 & \text{ for } x \le 0 \end{cases}$$

is $\tilde{h}(p) = (ip + \alpha)^{-1}$, and to use partial fractions.

(c) Find a solution y(x) to the differential equation in the case that

$$f(x) = \begin{cases} 1 & \text{for } x > 0\\ 0 & \text{for } x \le 0. \end{cases}$$