Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH1071-WE01

Title:
Linear Algebra I

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:	

Q1 In this question, make sure you explain your working.

1.1 Find C, the set of vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ which make an angle of $\pi/4$ with the line

$$L = \left\{ \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

Hence find the equation of the curve H along which C meets the plane $z = 1$.

1.2 Suppose $\mathbf{a}, \mathbf{d}$ and $\mathbf{u}$ are linearly independent vectors in $\mathbb{R}^3$ and write K for the line $\{\mathbf{a} + t\mathbf{d} : t \in \mathbb{R}\}$. We define the following subsets of $\mathbb{R}^3$:

$X_1 = \{\mathbf{x} : \mathbf{u} \times \mathbf{x} = \mathbf{u} \times \mathbf{a}\}$;

$X_2 = \{\mathbf{x}$: there is some $\mathbf{w} \in K$ such that $\mathbf{u} \times \mathbf{x} = \mathbf{u} \times \mathbf{w}\}$;

$X_3 = \{\mathbf{x}$: there is some $\mathbf{w} \in K$ such that $\mathbf{u} \times \mathbf{x} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \cdot \mathbf{x} = \mathbf{u} \cdot \mathbf{w}\}$.

For each of the above sets, describe the set of points defined.

[*Hint: do not use coordinates; think about the properties of the cross product instead. For X_1 consider rewriting the equation as* $\mathbf{u} \times \mathbf{x} - \mathbf{u} \times \mathbf{a} = \mathbf{0}$.]

Q2 (a) Use Gauss-Jordan to find all solutions of the set of equations

$$\begin{array}{rrrrl} x & +2y & +6z & & = 1 \\ & 2y & +3z & +w & = 2 \\ 2x & -y & +z & +w & = 2 \\ -x & & -2z & & = 1\,. \end{array}$$

Make sure you show your working.

(b) Now consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 6 & 0 \\ 0 & 2 & 3 & 1 \\ 2 & -1 & 1 & 1 \\ -1 & 0 & -2 & 0 \end{pmatrix}.$$

Use your calculations from (a) to find the nullspace of A.

Q3 (a) Let D be the linear map $\mathbb{R}[x]_3 \to \mathbb{R}[x]_3$ given by

$$D(p(x)) = 2p(x) + \frac{d}{dx}p(x) - x\frac{d^2}{dx^2}p(x)\,.$$

Using the standard (ordered) basis of $\mathbb{R}[x]_3$, namely $\{1, x, x^2, x^3\}$, find A_D, the matrix representative of D.

(b) Now find A_D^{-1}, showing your working. Hence or otherwise find polynomials $p_0(x)$, $p_1(x)$, $p_2(x)$, $p_3(x) \in \mathbb{R}[x]_3$ which satisfy $D(p_r(x)) = x^r$.

Q4 Which of the following are true statements, and which are false? Justify your answer by proving those you think are true, and giving a counter-example for those you think are false.

4.1 The function $F\colon \mathbb{R}[x]_n \to \mathbb{R}[x]_n$ given by $F(p(x)) = p^2(x)$ is a linear map.

4.2 If $f\colon \mathbb{R}^4 \to \mathbb{R}^3$ and $g\colon \mathbb{R}^3 \to \mathbb{R}^4$ are linear maps, then the composite $gf\colon \mathbb{R}^4 \to \mathbb{R}^4$ defined by $(gf)(\mathbf{v}) = g(f(\mathbf{v}))$ is never an isomorphism. [*Hint: you may wish to consider the Rank-Nullity theorem applied to one of f or g.*]

4.3 If U and W are vector subspaces of a vector space V, and if $U \cap W$ contains only the zero vector, then any vector $\mathbf{v}$ in $U + W$ has only one way of being written in the form $\mathbf{v} = \mathbf{u} + \mathbf{w}$ where $\mathbf{u} \in U$ and $\mathbf{w} \in W$.

Q5 In this question you may use any results from lectures you need provided you state them carefully.

5.1 Prove that the three vectors

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \right\}$$

form a basis of $\mathbb{R}^3$. Use this to prove that the vectors

$$\left\{ \begin{pmatrix} a \\ 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} b \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} c \\ 2 \\ 2 \\ 3 \end{pmatrix} \right\}$$

are linearly independent in $\mathbb{R}^4$ whatever values a, b and c take.

5.2 Suppose $\{\mathbf{e}_1, \ldots, \mathbf{e}_5\}$ is the standard basis for $\mathbb{R}^5$, and that $\{\mathbf{w}_1, \mathbf{w}_2\}$ is a linearly independent pair of vectors in the span of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$.

The Steinitz Exchange Theorem is used to construct a new basis of $\mathbb{R}^5$ consisting of $\mathbf{w}_1$ and $\mathbf{w}_2$ along with three of the original basis vectors $\mathbf{e}_r$. Depending on the initial vectors $\mathbf{w}_1$ and $\mathbf{w}_2$, and the subsequent choices made when applying the Theorem, it might be possible to construct more than one such basis this way.

Briefly explaining your reasoning, say which of the following statements can be true, and which statements are never true. For those you think can be true, give an example of $\mathbf{w}_1$ and $\mathbf{w}_2$ which illustrate that statement.

(i) It is only possible to construct one such basis.
(ii) It is possible to construct exactly two bases.
(iii) It is possible to construct exactly three bases.
(iv) It is possible to construct exactly four bases.
(v) It is possible to construct five or more bases.

Q6 Find the solution to the system of ordinary differential equations

$$\dot{x}(t) = 3x(t) - y(t)\,,$$
$$\dot{y}(t) = -x(t) + 3y(t)\,,$$

subject to the initial conditions $x(0) = 2, y(0) = 4$. Remember that $\dot{x}(t) = dx(t)/dt$.

Q7 Let $V = \mathbb{R}[x]_2$ be the vector space of real polynomials of degree at most two and consider the family of linear operators $\mathcal{L}_a : V \to V$ defined by

$$\mathcal{L}_a(p(x)) = a\,xp'(x) + a\,p(x+1) + \frac{1}{x}\int_0^x p(y)dy\,,$$

with $p(x) \in V, p'(x) = dp(x)/dx$, and $a \in \mathbb{R}$.

(a) Define $S \subseteq \mathbb{R}$ as the set of all values $a \in \mathbb{R}$ for which the linear operator $\mathcal{L}_a$ has an eigenvalue equal to 1. Find the set S.

(b) Fix $a_\star = \max_{a\in S} a$, i.e. the maximal value $a_\star$ for which $\mathcal{L}_{a_\star}$ has an eigenvalue equal to 1. Find the eigenfunction of $\mathcal{L}_{a_\star}$ with eigenvalue 1.

Q8 **8.1** Consider $V = \mathbb{R}^2$ and determine the value of $b \in \mathbb{R}$ for which the expression

$$(\mathbf{x}, \mathbf{y}) = x_1 y_1 + b\,x_1 y_2 + 2x_2 y_1 + 8x_2 y_2\,,$$

defines a real inner product on V, where $\mathbf{x}, \mathbf{y} \in V$. Using the inner product just found, compute the angle between the vectors

$$\mathbf{u} = \frac{1}{10}\begin{pmatrix}12\\1\end{pmatrix}\,, \qquad \mathbf{v} = \begin{pmatrix}-1\\2\end{pmatrix}\,.$$

8.2 Let $V = \mathbb{R}[x]_2$ be the vector space of real polynomials of degree at most 2, and consider the inner product

$$(p, q) = \int_{-1}^{1} p(x)q(x)dx\,,$$

with $p(x), q(x) \in V$. Find a basis for the orthogonal complement, $U^\perp$, of the vector subspace $U = \text{span}\{x^2 + 1\}$ in V.

Q9 **9.1** Consider the vector space $M_n(\mathbb{C})$ of $n \times n$ complex matrices and define the linear transformation $\mathcal{A} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$

$$\mathcal{A}(B) = B^* = (\overline{B})^t\,,$$

with $B \in M_n(\mathbb{C})$. Find the possible eigenvalues for $\mathcal{A}$.

9.2 Consider again the vector space $M_n(\mathbb{C})$ and define the linear transformation $\mathcal{T} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$

$$\mathcal{T}(B) = B^t\,,$$

with $B \in M_n(\mathbb{C})$. Show that the linear operator $\mathcal{T}$ is hermitian with respect to the inner product on $M_n(\mathbb{C})$ defined by $\langle A, B\rangle = \text{Tr}(A\,B^*)$, i.e. show that $\langle \mathcal{T}(A), B\rangle = \langle A, \mathcal{T}(B)\rangle$.

Q10 Consider the set of matrices

$$G = \left\{ g \in M_4(\mathbb{R}) \,,\; g = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix}, \text{ with } A, B \in H \right\} \,,$$

where **0** denotes the 2×2 zero matrix and H denotes a group under matrix multiplication of 2×2 matrices. Show that G forms a group under matrix multiplication. You may assume associativity.

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