Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH1081-WE01

Title:

Calculus I (Maths Hons)

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

Q1 **1.1** Evaluate the following limit or explain why it does not exist.

$$\lim_{x \to 0} \cos\left(e^{1/x^2}\right) \sin x \ .$$

You may use any valid method.

1.2 Evaluate the following limit or explain why it does not exist.

$$\lim_{x \to 0} \frac{x \sin(\sinh x)}{e^{x^2} - 1} \ .$$

You may use any valid method.

1.3 Calculate $f'(\pi/4)$ where

$$f(\theta) = \int_{\sin^2 \theta}^{\cos^2 \theta} e^{-x^2} dx \ .$$

Q2 (a) Find a recursion relation for

$$I_n = \int_{-\pi}^{\pi} x^n \cos x dx \ .$$

(b) Evaluate I_6 and I_{13}.

Q3 Use a change of variables

$$x = \frac{1}{2} u \sin v \ , \quad y = \frac{1}{3} u \cos v$$

to evaluate

$$\int \int_D \frac{xy}{4x^2 + 9y^2} dx dy$$

where D is the part of the interior of the curve $4x^2 + 9y^2 = 25$ with $x > 0$ and $y > 0$.

Q4 **4.1** Find the function $g(x)$ satisfying

$$g'(x) = g(x) + e^{2x} \ , \quad g(0) = 3 \ .$$

4.2 Calculate the Taylor polynomial of degree 2 about $x = \log \pi$ for the function $f(x) = \cos(e^x)$.

Q5 (a) Find the Fourier series for the function f with period 4 which is defined by

$$f(x) = \begin{cases} 0 & \text{for } 1 \leq |x| \leq 2 \\ 1 - |x| & \text{for } |x| < 1 \end{cases} \ .$$

(b) Use the Fourier series from part (a) to evaluate

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \ .$$

Q6 **6.1** A function is defined to be

$$f(x, y) = 100 - \frac{1}{15}\left(2x^2 + 3xy + y^2\right).$$

Find all unit vectors $\hat{\mathbf{n}}$ such that the directional derivative of f in the direction of $\hat{\mathbf{n}}$ at the point $(x, y) = (2, -3)$ is equal to $1/25$.

6.2 Find and classify the stationary points of the function

$$f(x, y, z) = 2x^3 + 2x^2 + y^2 + z^2 - 2xy - 4xz.$$

Q7 A change of coordinates is given by

$$u(x, y) = \sqrt{x} + \sqrt{y}$$
$$v(x, y) = \sqrt{x} - \sqrt{y},$$

where x and y are both positive.

(a) Use the Chain Rule to express f_u, f_v and f_{uv} in terms of x and y.

(b) Using your answer to part a) or otherwise, show that

$$f(x, y) = F(\sqrt{x} + \sqrt{y}) + G(\sqrt{x} - \sqrt{y}),$$

where F and G are arbitrary functions, is a solution to the PDE

$$\frac{f_{xx}}{y} - \frac{f_{yy}}{x} + \frac{f_x - f_y}{2xy} = 0.$$

Q8 Consider the ordinary differential equation

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - \lambda y = 0.$$

(a) Explain what is meant by a *regular point* of an ODE. Which points are regular for this ODE?

(b) A solution to the equation can be expressed in the form $y = \sum_{n=0}^{\infty} a_n x^n$. Find a recurrence relation for the coefficients a_n.

(c) In the case that $\lambda = 4$, find a formula for the 'even' coefficients a_{2n} in terms of a_0, and use your answer to find an explicit solution (i.e. not as a series) to the differential equation for this value of λ. Hint: it may help to consider the Taylor series for the exponential function.

Q9 A bar of metal lies along the x-axis between $x = 0$ and $x = \pi$. Its temperature $u(x, t)$ obeys the heat equation $u_t = k^2 u_{xx}$, and its ends are held at zero degrees so that $u(0, t) = u(\pi, t) = 0$.

(a) Show that the function

$$u(x, t) = \sum_{n=1}^{\infty} A_n \sin(nx) e^{-k^2 n^2 t}$$

satisfies the heat equation and the boundary conditions at $x = 0$ and $x = \pi$.

(b) In the case that the bar initially has temperature

$$u(x, 0) = f(x) = \begin{cases} 100 \text{ for } 0 < x \leq \pi/2 \\ 0 \text{ for } \pi/2 < x < \pi \end{cases}$$

find the coefficients A_n and hence the solution $u(x, t)$ for $t > 0$.

(c) The total heat in the metal bar is given as $H(t) = \int_0^{\pi} u(x, t) dx$. Given the initial condition in part b), find an estimate for the time required for the bar to lose 90% of the initial heat it has at $t = 0$ in terms of k.

Q10 The function $y(x)$ satisfies the differential equation

$$\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + 2y = f(x).$$

(a) By taking the Fourier transform of this equation with respect to x, find an expression for $\tilde{y}(p)$, the Fourier transform of $y(x)$ in terms of $\tilde{f}(p)$, the Fourier transform of $f(x)$.

(b) Using your result for part a), find an expression for $y(x)$ in the form of a convolution

$$y(x) = f(x) \star G(x)$$

where you should determine the function $G(x)$ explicitly. It may help to use the fact that the Fourier transform of the one sided exponential

$$h(x) = \begin{cases} e^{-\alpha x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

is $\tilde{h}(p) = (ip + \alpha)^{-1}$, and to use partial fractions.

(c) Find a solution $y(x)$ to the differential equation in the case that

$$f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases}$$