Durham University

# EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
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| May/June | 2023 | MATH1551-WE01 |

| Title: |
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| Maths For Engineers and Scientists |

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| Time: | 3 hours | |
| Additional Material provided: | Formula sheet. | |
| Materials Permitted: | | |
| Calculators Permitted: | No | Models Permitted: Use of electronic calculators is forbidden. |

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| Instructions to Candidates: | Credit will be given for your answers to each question.<br>All questions carry the same marks.<br>Students must use the mathematics specific answer book. |

| Revision: | |
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ED01/2023

**Q1** Consider the system of equations

$$\begin{aligned} 3x + 2y - z &= 12 \\ 2x + 4y - z &= 8 \\ -x - y + 2z &= 9. \end{aligned}$$

(a) (i) Is the Jacobi method guaranteed to converge for this system for any given initial values? State your reasoning.

(ii) Is the Gauss-Seidel method guaranteed to converge for this system for any given initial values? State your reasoning.

(b) Write down the Gauss-Seidel iteration scheme for this system and perform one iteration starting from initial values $(x^{(0)}, y^{(0)}, z^{(0)}) = (1, 1, 2)$.

(c) Use Gaussian elimination to find the exact solution of the system of equations.

**Q2** (a) Find the modulus and principal argument of all possible values of $(1 - i)^{2i}$.

(b) Find all complex solutions of the equation $\tan(2z) = 3i$, expressing your answers in the form $z = x + iy$ where $x, y \in \mathbb{R}$.

(c) (i) Show that if $z = e^{i\theta}$, then

$$z^n + \frac{1}{z^n} = 2\cos(n\theta).$$

(ii) Given $z + \frac{1}{z} = \sqrt{2}$, use the result in part (i) to determine the value of $z^8 + \frac{1}{z^8}$.

**Q3** (a) Consider the following set of vectors in $\mathbb{R}^3$: $S = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$.

(i) Is $S$ a basis for $\mathbb{R}^3$? Justify your answer.

(ii) Suppose each of the vectors in $S$ gives the position of a corner of a triangle. Find a Cartesian equation for the plane that this triangle lies in.

(iii) Which point on this plane is the closest to the origin?

(b) (i) Find Cartesian equations of the line that has parametric equation $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$, $t \in \mathbb{R}$.

(ii) What angle does this line make with the line that has Cartesian equations

$$\frac{x - 1}{4} = \frac{y - 2}{4} = \frac{z + 1}{2}?$$

Give your answer in terms of $\pi$.

**Q4** (a) Calculate the eigenvalues and corresponding eigenspaces of the matrix

$$A = \begin{pmatrix} -1 & -5 & -4 \\ 0 & -3 & 0 \\ -2 & 5 & 1 \end{pmatrix}$$

(b) From your answer to part (a), do you expect $A$ to be diagonalisable? Explain your reasoning.

**Q5** (a) Find each of the following limits or argue why they don't exist:

$$\text{(i)} \lim_{x\to 0} (x+1)^{2023}, \qquad \text{(ii)} \lim_{x\to 0} \frac{5x^3+8x^2}{3x^4-16x^2}, \qquad \text{(iii)} \lim_{x\to 0} \frac{\cosh(x)-1}{x^2}.$$

(b) *Without* using l'Hôpital's Rule, use the limit definition of differentiation to find $f'(1)$ for $f(x) = \dfrac{1}{\sqrt{3x-2}}$.

**Q6** (a) Find the degree 3 Taylor polynomial of $\tan(x)\ln(1+x)$ about $x = 0$.

(b) Consider the equation $e^{-x} = \sin x$.

(i) Use an appropriate theorem to show that there is at least one solution $x_*$ to this equation in the interval $0 < x < \dfrac{\pi}{2}$.

(ii) Write out a Newton-Raphson iteration formula (for this equation) that would converge to $x_*$ for a close enough initial guess $x_0$.

**Q7** The elevation of a surface is specified by the function

$$f(x, y) = 2x^3 + 2y^3 - 9x^2 + 3y^2 - 12y.$$

(a) Determine the critical points of $f(x, y)$ and identify each as a local minimum, local maximum or saddle point.

(b) Find the directional derivative of $f(x, y)$ in the direction $\mathbf{i} - 2\mathbf{j}$ at the point $(0, 0)$.

(c) An ant can walk on the surface, and is standing at the point $(0, 0)$. In which direction should the ant walk in order for their elevation to decrease most rapidly?

**Q8** (a) Find the general solution of the first-order differential equation

$$2x\frac{dy}{dx} + y = 2x^{5/2}.$$

(b) Find the specific solution of the second-order differential equation

$$y'' + 16y = 16\cos(4x)$$

satisfying the initial conditions $y(0) = 1$ and $y'(0) = 1$.