Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH1561-WE01

Title:
Single Mathematics A

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

Q1 (i) Use the derivative of $\sinh x$ to find $\dfrac{d}{dx}(\operatorname{arcsinh} x)$.

(ii) Express the complex number $\left(\dfrac{2+i}{1+2i}\right)^2$ in the form $a+ib$ with a and b real.

(iii) Find $\dfrac{d}{dx}((\sin x)^{\sin x})$.

Q2 (i) Evaluate the definite integral

$$\int_0^{\pi/2} \cos x \sin^2 x \, dx \, .$$

(ii) Find an expression for the indefinite integral $\int x^n \sinh x \, dx$ in terms of $\int x^{n-2} \sinh x \, dx$ and use this to find $\int x^2 \sinh x \, dx$ and $\int x^4 \sinh x \, dx$.

(iii) Evaluate the indefinite integral

$$\int \frac{1}{x^3-1} \, dx \, .$$

Hint: $\displaystyle\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \arctan(x/a)$

Q3 (i) Evaluate the following limits, you may use any method. You will only get full marks if you explain carefully the steps and rules you are using to prove the limits:

(a) $\displaystyle\lim_{x \to \pi/2} \frac{\cos x}{x - \pi/2}$

(b) $\displaystyle\lim_{x \to 2/\pi} x \cos \tfrac{1}{x}$

(c) $\displaystyle\lim_{x \to 0} x \cos \tfrac{1}{x}$.

(ii) (a) Find all complex solutions of the equation

$$(z-1)^4 = 1$$

in the form $z = x + iy$.

(b) Plot these solutions on the complex plane.

(c) Hence or otherwise completely factorise the polynomial $(z-1)^4 - 1$ over the complex numbers.

(d) Completely factorise the same polynomial $(x-1)^4 - 1$ over the real numbers.

Q4 (a) Determine whether or not the series

$$\sum_{n=1}^{\infty} \frac{6n}{\sqrt{n^4+2}}, \qquad \sum_{n=1}^{\infty} \frac{1}{n(\ln 2n)^{3/2}}$$

converge.

(b) Determine the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{n} x^n.$$

Determine whether the series converges at the endpoints of the interval of convergence.

Q5 **5.1** For the function $f(x) = \exp\left(\frac{1+\sin x}{2}\right)$,

(a) Find the second-order Taylor polynomial $p_2(x)$ about $x = 0$.

(b) Use the Lagrange form of the remainder to obtain a bound on the error $f(x) - p_2(x)$.

5.2 (a) Consider the matrix

$$A = \begin{pmatrix} a & 0 & \frac{1}{\sqrt{2}} \\ 0 & b & c \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Find the values of a, b and c such that A is orthogonal. Assume all of a, b and c are positive.

(b) Use the fact that A is orthogonal to write down the inverse of A.

(c) Write down the property of the eigenvalues of A that results from A being orthogonal.

Q6 For which values of $a \in \mathbb{R}$ does the system of linear equations

$$\begin{aligned} (a^2-1)x + \ \ (a-1)y - az &= -1 \\ (a^2-1)x + 2(a-1)y - \ \ z &= -1 \\ (a-1)y + \ \ z &= \ \ a \end{aligned}$$

have (a) no solutions, (b) a unique solution, (c) infinitely many solutions?

Find the solutions in cases (b) and (c) and, in case (c), also say whether the solution represents a line or a plane, and write down the solution of the corresponding homogeneous equation.

Q7 **7.1** Find the eigenvalues and corresponding eigenvectors of the matrix

$$B = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & -i & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

7.2 Prove that a hermitian matrix must have real eigenvalues.