Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH1571-WE01

Title:
Single Mathematics B

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

Q1 Consider two three-dimensional lines $\mathbf{r}_1, \mathbf{r}_2$, given by the equations:

$$\mathbf{r}_1 = \mathbf{a}_1 + t_1\mathbf{b}_1, \quad \mathbf{r}_2 = \mathbf{a}_2 + t_2\mathbf{b}_2.$$

Where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1$ and $\mathbf{b}_2$ are constant vectors.

(a) The minimum distance d_m between the two lines is given by the following:

$$d_m = \frac{|(\mathbf{a}_1 - \mathbf{a}_2)\cdot(\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}.$$

Calculate this distance for the following specific cases:

$$\mathbf{a}_1 = \mathbf{i} + 3\mathbf{j} + 5\mathbf{k}, \qquad \mathbf{a}_2 = 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k},$$
$$\mathbf{b}_1 = \mathbf{i} + 2\mathbf{k}, \qquad \mathbf{b}_2 = 2\mathbf{j} + 3\mathbf{k}.$$

(b) Find the Fourier series for the following function, defined in the interval $(-\pi, \pi)$:

$$f(x) = 1 \quad (-\pi < x < 0), \qquad f(x) = \cos(x) \quad (0 < x < \pi).$$

You may use the identities

$$\cos(nx)\sin(mx) = \frac{1}{2}\left(\sin((m-n)x) + \sin((m+n)x)\right),$$
$$\cos(nx)\cos(mx) = \frac{1}{2}\left(\cos((m-n)x) + \cos((m+n)x)\right).$$

Q2 Consider a time-dependent interacting population model for scalar populations $x(t), y(t)$ in the following form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by + f(t),$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy + g(t),$$

where $a, d > 0$ are positive constants, b, c are real constants, and $f(t), g(t)$ smooth scalar functions.

(a) Describe the effect on the populations x, y by all the terms on the right hand side of this system of equations, giving particular attention to how these effects might vary over time.

(b) Show that one can reduce the system to a second order ordinary differential equation for y in the following form:

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + A\frac{\mathrm{d}y}{\mathrm{d}t} + By = C(t). \tag{1}$$

Where A, B are constants and $C(t)$ a scalar function, all of whose explicit forms you must state in terms of a, b, c, d, f, g.

(c) Find the general solution to (1) for the case $A = 0, B = 0$ and

$$C(t) = e^{\tan(t)}\sec^4(t) + 2e^{\tan(t)}\tan(t)\sec^2(t).$$

Hint:
$$\frac{\mathrm{d}\tan(t)}{\mathrm{d}t} = \sec^2(t), \quad \frac{\mathrm{d}\sec(t)}{\mathrm{d}t} = \sec(t)\tan(t).$$

Q3 Consider a charged particle (of charge e) that is acted on by the electromagnetic force:

$$\mathbf{F}_{eq} = e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right),$$

where $\mathbf{E}$ is an electric field, $\mathbf{v}$ the particle velocity and $\mathbf{B}$ a magnetic field. Further assume this is the only force acting on the particle.

(a) Assume an electric field in the form $\mathbf{E} = \cos(\omega t)\mathbf{i}$ and that the particle moves at a **constant** velocity $\mathbf{v} = v\,\mathbf{j}$, *i.e.* it is in equilibrium. Calculate $\mathbf{B}$ and describe the relative motion of the electric and magnetic fields

(b) Now consider a radial electric field $\mathbf{E} = mE_0\mathbf{e}_r$ and a circumferential magnetic field $\mathbf{B} = mB_0\mathbf{e}_\theta$. Here both E_0 and B_0 are constant and $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ is the cylindrical polar basis. The particle's acceleration is purely radial: $\mathbf{a} = e\,\mathbf{e}_r$. Find the general form of the path $\mathbf{r}$ of the particle, describe it qualitatively and explain why the particle must be negatively charged!

You may need to use the following relationships:

$$\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_z, \quad \mathbf{e}_r \times \mathbf{e}_z = -\mathbf{e}_\theta, \quad \mathbf{e}_\theta \times \mathbf{e}_z = \mathbf{e}_r,$$

and

$$\frac{\mathrm{d}\mathbf{e}_r}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\mathbf{e}_\theta, \quad \frac{\mathrm{d}\mathbf{e}_\theta}{\mathrm{d}t} = -\frac{\mathrm{d}\theta}{\mathrm{d}t}\mathbf{e}_r.$$

Q4 (a) Consider the following differential:

$$df = x^2y^2\;dx + cx^3y\;dy\;.$$

For which values of c is the differential exact?

(b) Find and classify the critical points of the following function:

$$f(x,y) = \sin(x)y\;.$$

(c) Consider the function:

$$f(x,y) = -\left((x-3)^2 + y^2\right)^4\;.$$

Find its critical points. Is the usual second derivative criterion sufficient to classify the critical points? Explain why (or why not) and classify the critical points.

Q5 (a) Imagine you are flying a plane through a three dimensional smoke cloud. The density of smoke at a point (x, y, z) is given by:

$$\rho(x, y, z) = \exp(-2z)(\sin(x) + 2y^2)$$

(i) At the point $(x, y, z) = (\frac{\pi}{2}, 2, 3)$, in which direction should you fly so that the density decreases as quickly as possible?

(ii) The level surfaces of the density are defined to be two-dimensional surfaces along which $\rho(x, y, z) = \rho_0$, where ρ_0 is a constant. At the point $(x, y, z) = (\frac{\pi}{2}, 2, 3)$, find two linearly independent vectors that are tangent to this level surface. (Here "linearly independent" means that the second vector should not be a multiple of the first.)

(b) Compute the curl of the following vector field:

$$\mathbf{V}(x, y) = 2xy\mathbf{i} + 13yz\mathbf{j} + \sin(x)\mathbf{k} \ .$$

(c) Compute the divergence of the following vector field, which is expressed in polar coordinates (r, θ) on $\mathbb{R}^2$:

$$\mathbf{W}(r, \theta) = 12r^2\hat{\mathbf{e}}_{\mathbf{r}}$$

Hint: one way to do this is to convert it back to Cartesian (x, y) coordinates and use the regular formula for divergence from there.

Q6 (a) Find the general solution to the partial differential equation for $u(x, y)$:

$$u_{xx} - 2yu = 0 \ .$$

Find also the particular solution such that $u(x = 0, y) = y^2$ and $u(x, y) \to 0$ as $x \to \infty$.

(b) Consider the following modified version of the diffusion equation for $u = u(x, t)$:

$$u_{xx} - \frac{1}{\alpha^2}u_t - \beta u = 0 \ . \tag{2}$$

Use the method of separation of variables to write $u(x, t)$ in the following separated form:

$$u(x, t) = X(x)T(t)$$

and obtain a linked pair of ODEs for $X(x)$ and $T(t)$. Use this to show that the partial diferential equation Eq. (2) above admits solutions of the form:

$$u(t, x) = \sin(kx)\exp(-\gamma t)$$

where γ depends on k, β, and α in a way that you should determine.

Q7 The Eurovision Song Contest is an annual singing competition. After an initial (boring) selection, 20 acts compete in the televised (fun) final of the competition.

7.1 Paul, Maciej, Chris, Nabil, and Clare plan to watch the final together. They decide that they will each randomly choose one act to support.

(i) How many different ways are there to allocate the acts?

(ii) The acts from France, Germany, Italy, Spain, and the UK are known as the "Big Five", and always qualify. What is the probability that at least four of the "Big Five" acts are chosen?

7.2 During the song contest, Clare decides to count how many times different "Eurovision stereotypes" happen. For example, based on past experience, she knows that key changes in songs seem to happen randomly, at a steady rate of about one every ten minutes during the program.

(i) Which distribution can we use to model the number of key changes that happen in one hour of the contest?

(ii) What is the probability that at least five key changes happen in one hour?

(iii) Chris decides to count the number of times a smoke machine is used in the contest. Assuming that the smoke machine is used randomly, at a steady rate of four times per hour, and that this happens **independently** of key changes, what is the distribution of the **total** number of key changes and smoke-machine-uses in one hour?