Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH1597-WE01

Title: Probability I

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Students must use the mathematics specific answer book.

Revision:	

Q1 Jimmy the gambler charges a £2 stake for a player to take a turn at a dice game. If a player pays the stake, then they roll five standard fair six-sided dice to obtain a *hand*. Jimmy awards the player winnings of £2 if the hand is a 'pair', and pays £4 if the hand is a 'double pair'; otherwise, the player gets no winnings.

A *pair* is a hand in which precisely two dice show the same number, and the three other dice all show numbers that are all different from each other, and different from that of the pair; for example 12446 is a pair, but 14446 is not, 11446 is not, and neither is 11444.

A *double pair* is an outcome where two dice show the same number, two further dice show another number (different from that of the first pair), and the remaining die shows yet another number different from both pairs; for example 11446 is a double pair, but 11444 is not.

Calculate a player's expected profit (i.e., winnings minus stake) if they decide to play a turn of Jimmy's game.

Q2 Edward is employed as quality control officer for a producer of jars of honey. If the production process is running well, then the probability that a randomly selected jar is of good quality is 4/5, independently of the quality of any of the other jars. If the process is not running well, then the probability that a jar is good is only 2/5. Without further information, the probability that the process is running well is known to be 3/4.

(a) A sample of 4 jars is taken randomly from the latest batch. What is the probability that all jars are of good quality?

(b) Edward tests the first 3 jars, and finds that two are good and one is not. What is the probability that the process is running well?

(c) Edward then tests the 4th jar from the sample. What is the (conditional) probability that it is good?

Q3 A fair coin is tossed n times independently, where n is a positive integer. Fix some $k \in \{0, 1, \ldots, n\}$. Let

$$X = \text{number of heads in the first } k \text{ tosses;}$$
$$Y = \text{number of heads in the last } n - k \text{ tosses.}$$

Also set $T = X + Y$, the total number of heads in the n tosses.

(a) Calculate $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$.

(b) Explain whether or not X and Y are independent.

(c) Calculate $\mathbb{E}(XY)$.

(d) Calculate $\mathbb{E}(XT)$. Hence find $\mathbb{C}\mathrm{ov}(X, T)$.

Q4 Jointly continuous random variables X, Y have joint probability density function

$$f(x,y) = \begin{cases} \theta(x^2 - y^2) & \text{for } 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Calculate the value of the constant θ.

(b) Calculate the marginal probability density functions f_X and f_Y of X and Y, respectively.

(c) Are X and Y independent? Explain.

(d) Calculate $\mathbb{P}(Y < X^2)$ and $\mathbb{P}(Y > 1/2 \mid X > 1/2)$.

Q5 A discrete random variable X is said to have a *geometric distribution* with parameter $q \in (0, 1]$, written $X \sim \text{Geo}(q)$, if it has probability mass function

$$\mathbb{P}(X = x) = (1-q)^{x-1}q, \text{ for } x \in \{1, 2, 3, \ldots\}.$$

In your answers to the questions below, you may use any standard properties of moment generating functions that you state clearly and correctly.

(a) Show that the moment generating function of $X \sim \text{Geo}(q)$ satisfies

$$M_X(t) := \mathbb{E}(\mathrm{e}^{tX}) = \frac{q\mathrm{e}^t}{1-(1-q)\mathrm{e}^t}, \text{ for } t < -\log(1-q).$$

(b) Use M_X to calculate $\mathbb{E}(X)$ and $\mathbb{V}\text{ar}(X)$.

(c) Consider independent random variables $X_1 \sim \text{Geo}(1/2)$ and $X_2 \sim \text{Geo}(1/4)$, and let $Y := 4X_1 + 2X_2$. Does Y have a geometric distribution? Explain.