Durham University

EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
| --- | --- | --- |
| May/June | 2023 | MATH2051-WE01 |

| Title: |
| --- |
| Numerical Analysis II |

| | | |
| --- | --- | --- |
| Time: | 3 hours | |
| Additional Material provided: | | |
| Materials Permitted: | | |
| Calculators Permitted: | Yes | Models Permitted: Casio FX83 series or FX85 series. |

| | |
| --- | --- |
| Instructions to Candidates: | Answer all questions.<br>Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.<br>Students must use the mathematics specific answer book. |

| Revision: | |
| --- | --- |

ED01/2023

## SECTION A

**Q1** (a) Define what is meant by a floating-point number having a *finite precision* and a *finite range*.

(b) Starting with an interval $[a, b]$, describe briefly the *bisection method* to find an approximate solution of $f(x) = 0$.

(c) Using the bisection method and some floating-point arithmetic to find the zeros of $f(x) = \sin x$, what are the *best absolute errors* one can expect for $x_0 = 0$ and $x_1 = \pi$? Justify your answers.

**Q2** (a) Compute the error $E(h; x)$ in the backward differentiation formula,

$$f'(x) = \frac{3f(x) - 4f(x-h) + f(x-2h)}{2h} + E(h; x).$$

(b) Given $f(x)$, $f(x - h/2)$, $f(x - h)$ and $f(x - 2h)$, construct a higher-order approximation to $f'(x)$. Justify the order of your approximation, but no need to compute the error term explicitly.

**Q3** (a) Decompose the matrix (without pivoting)

$$A = \begin{pmatrix} -6 & 0 & -7 & 2 \\ 12 & 6 & 8 & -4 \\ 48 & -48 & 97 & -23 \\ -42 & -36 & -34 & -4 \end{pmatrix}$$

into a product $LU$ where $L$ is unit lower triangular and $U$ upper triangular.

(b) Use this to solve the linear system $Ax = b$ where $b = (2, 3, 5, 7)$.

(c) State briefly why one might prefer to use $LU$ decomposition rather than computing $A^{-1}$ to solve $Ax = b$.

**Q4** Defining $T_n(x) := \cos(n \cos^{-1} x)$ for $n \in \{0, 1, \cdots\}$ and $x \in [-1, 1]$, prove the following:

(a) $T_n(1) = 1$ and $T_n(-1) = (-1)^n$ for $n \in \{0, 1, \cdots\}$;

(b) $T_n(x)T_k(x) = \frac{1}{2}[T_{n+k}(x) + T_{n-k}(x)]$ for $k,\, n \in \{0, 1, \cdots\}$;

(c) that $T_n(x)$ is a polynomial of degree $n$ for $x \in [-1, 1]$;

(d) with $c_0 = \pi$ and $c_n = \pi/2$ for $n \in \{1, 2, \cdots\}$, the identity

$$\int_{-1}^{1} T_k(x)T_n(x)\frac{\mathrm{d}x}{\sqrt{1-x^2}} = c_n\delta_{kn}.$$

## SECTION B

**Q5** Given $b > a$ and $f \in C^5([a,b])$, we seek $p \in \mathcal{P}_4$ such that $p(a) = f(a)$, $p'(a) = f'(a)$, $p(b) = f(b)$, $p'(b) = f'(b)$ and $p''(b) = f''(b)$.

(a) Prove that any such $p$, if it exists, must satisfy

$$f(x) - p(x) = \frac{f^{(5)}(\xi)}{5!}(x-a)^2(x-b)^3$$

for every $x \in [a,b]$ with $\xi = \xi(x)$.

(b) Prove that if such $p$ exists, it is unique.

(c) Prove the existence of $p$.

**Q6** (a) Write down the Newton–Raphson method to solve $f(x) = 0$.

(b) Define what is meant for a sequence $(x_n)$ to converge with order at least 2.

(c) Stating any relevant assumptions of $f$, prove that the Newton–Raphson method typically converges with order exactly 2.

(d) Let $f(x) = x^3 - 5$ and take $x_0 = 2$. Computing the first few Newton–Raphson iterates, estimate (to within $\pm 1$) the smallest $n$ such that $|f(x_n)| \le 10^{-1000}$.

**Q7** (a) Given a vector norm $\|\cdot\|_*$, define what is meant by its *induced norm* for matrices.

(b) Define what is meant by the *condition number* $\kappa_*(\cdot)$, and explain briefly the significance of $\kappa_*$ when solving the linear system $Ax = b$.

(c) Define what is meant by two (vector) norms $\|\cdot\|_*$ and $\|\cdot\|_{**}$ being *equivalent*.

(d) Show explicitly that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are equivalent norms.

(e) Given two equivalent vector norms $\|\cdot\|_*$ and $\|\cdot\|_{**}$, are their respective induced norms also equivalent? Justify your response.

**Q8** (a) Define what is meant by an *interpolatory quadrature* formula $\mathcal{I}_n(f)$.

(b) Define what is meant by the *degree of exactness* of an interpolatory quadrature formula.

(c) With $[a,b] = [-1,1]$ and $x_* = \sqrt{3/7}$, determine the degree of exactness of

$$\mathcal{I}_4(f) = \frac{1}{90}\big[9f(-1) + 49f(-x_*) + 64f(0) + 49f(x_*) + 9f(1)\big].$$

(d) Given the closed Newton–Cotes formula

$$\mathcal{I}_2(f) = \frac{b-a}{6}\Big[f(a) + 4f\Big(\frac{a+b}{2}\Big) + f(b)\Big],$$

write down the formula for the composite quadrature $\mathcal{C}_{2,m}(f)$. Given that the degree of exactness of $\mathcal{I}_2$ is 3, how does an error bound of $\mathcal{C}_{2,m}(f)$ depend on $m$?

(e) Write down the composite quadrature derived from $\mathcal{I}_4$ above and obtain an error bound for it, clearly stating the dependence on $m$.