Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH2071-WE01

Title:
Mathematical Physics II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 Consider the Lagrangian

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \sin(q_1 + q_2)\,.$$

1.1 Is any of the coordinates q_1, q_2 ignorable?

1.2 Write down the Euler-Lagrange equations of motion for this system. You do not need to solve them.

1.3 The Lagrangian L is invariant under time translations, so the energy

$$E = \left(\sum_{i=1}^{2} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - L$$

should be conserved. Show, by explicitly taking time derivatives, that E is indeed conserved along physical paths.

1.4 Find another conserved quantity in this system by first identifying a transformation $q_i \to q_i + \epsilon a_i + \ldots$ leaving the Lagrangian invariant to first order in ϵ, and then constructing its associated Noether charge

$$Q = \sum_{i=1}^{2} a_i \frac{\partial L}{\partial \dot{q}_i}\,.$$

Q2 Assume that we have a Lagrangian of the form

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \cos(q_1) - 2\sin(q_1 q_2 + q_2^2)\,.$$

2.1 Write down the equations of motion for the system. You do not need to solve them.

2.2 Show that $q_1(t) = q_2(t) = 0$ is a solution of the equations of motion.

2.3 Construct an approximate Lagrangian L_{app} describing the behaviour of small displacements away from $q_1(t) = q_2(t) = 0$.
[**Hint:** You might want to use $\cos(\epsilon) = 1 - \frac{1}{2}\epsilon^2 + O(\epsilon^4)$ and $\sin(\epsilon) = \epsilon + O(\epsilon^3)$.]

2.4 Construct the general solution of the equations of motion that follow from L_{app}.

Q3 At time $t = 0$, the wave function for a particle in one dimension is given by the following expression:

$$\psi(t=0,x) = C\left(\frac{1}{\sqrt{3}}\psi_{E=1}(x) - e^{iq}\psi_{E=2}(x)\right).$$

Here, $\psi_{E=1}(x)$ and $\psi_{E=2}(x)$ are orthonormal energy eigenfunctions of the system. Both C and q are real constants.

3.1 Determine the constant C.

3.2 If we measure the energy of the particle at $t = 0$, what are the possible outcomes of this measurement, and what are the probabilities for each of these outcomes?

3.3 How does this wave function evolve in time? (In other words, determine $\psi(t, x)$ for arbitrary t.)

Q4 Consider the elementary position and momentum operators

$$\hat{x} = x\,, \quad \hat{p} = -i\hbar\frac{\partial}{\partial x}\,.$$

4.1 Compute the commutator $[\hat{p}^2\hat{x}, \hat{x}]$.

4.2 Show that the operator $\hat{q}$ defined by

$$\hat{q}\psi(x) = e^{ig(x)}\hat{p}\left(e^{-ig(x)}\psi(x)\right) = -i\hbar e^{-ig(x)}\frac{\partial}{\partial x}\left(e^{ig(x)}\psi(x)\right)$$

for an arbitrary function $g(x)$ satisfies $[\hat{x},\, \hat{q}] = i\hbar$.

SECTION B

Q5 Consider a Lagrangian density

$$\mathcal{L}(u, u_x, u_t, u_{xx}) = \frac{1}{2}u_x u_t - (u_x)^3 - \frac{1}{2}(u_{xx})^2$$

depending on the field $u(x,t)$, its first derivatives with respect to space and time $u_x \coloneqq \partial u/\partial x$ and $u_t \coloneqq \partial u/\partial t$, and the second space derivative $u_{xx} \coloneqq \partial^2 u/\partial x^2 = \partial u_x/\partial x$.

In this case the variational principle yields an Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial u_{xx}}\right) = 0$$

describing the motion of the field u. As usual u, u_x, u_t and u_{xx} are to be considered independent variables when taking derivatives in this expression.

5.1 Use the Euler-Lagrange equation just given to derive the explicit form of the equation of motion for u. Show that if you define $\phi \coloneqq u_x$, the equation of motion can be written as

$$\phi_t = \frac{\partial W(\phi, \phi_{xx})}{\partial x}$$

for some function W that you should find, with $\phi_t \coloneqq \partial\phi/\partial t$ and $\phi_{xx} = \partial^2\phi/\partial x^2$.

5.2 Show that shifts of u given by $u \to u + \epsilon$ are symmetries of the Lagrangian density $\mathcal{L}$. Find the Noether charge

$$Q = \int_{-\infty}^{\infty} a \frac{\partial \mathcal{L}}{\partial u_t}\, dx$$

associated to these shifts, where a is the generator of the transformation acting on u, and show by taking explicit time derivatives that it is indeed conserved. You may assume that u and all of its spatial derivatives vanish as you take the limit $x \to \pm\infty$.

5.3 The energy density in this system is defined to be

$$\mathcal{E} \coloneqq u_t \frac{\partial \mathcal{L}}{\partial u_t} - \mathcal{L}\,.$$

Use the equations of motion you found above to show that the time derivative of the total energy

$$E(a,b) = \int_a^b \mathcal{E}\, dx$$

contained in the interval (a,b) satisfies

$$\frac{dE(a,b)}{dt} = \big[F(\phi, \phi_x, \phi_{xx}, \phi_{xxx})\big]_a^b$$

for some function F of ϕ and its space derivatives ϕ_x, ϕ_{xx}, ϕ_{xxx} that you should find. (We define $\phi_x \coloneqq \partial\phi/\partial x$ and $\phi_{xxx} \coloneqq \partial^3\phi/\partial x^3$.)

Q6 Consider a system described by the a-dependent Lagrangian

$$L_a = -\sqrt{1 - \dot{q_1}^2 - a\dot{q_2}^2}\,.$$

6.1 Find all values of a for which rotations around $(q_1, q_2) = (0, 0)$ are symmetries, including the possibility that the Lagrangian changes up to a total time derivative of a function $F(q_1, q_2)$:

$$L_a \to L_a + \epsilon \frac{dF(q_1, q_2)}{dt} + O(\epsilon^2)\,.$$

6.2 Construct the generalised momenta p_1, p_2 associated to q_1 and q_2, and from there the Hamiltonian H_a of the system, for arbitrary a. [**Hint:** When expressing the velocities in terms of generalised momenta, you might want to first consider combinations of the form $c_1 p_1^2 + c_2 p_2^2$, for some appropriate constants c_1, c_2.]

6.3 Define Q to be the Noether charge associated with rotations around the origin. Show that the Poisson bracket $\{H_a, Q\}$ vanishes if and only if a is such that rotations around $(q_1, q_2) = (0, 0)$ are symmetries.

Q7 This problem deals with the unit-mass harmonic oscillator, which has a Hamiltonian

$$\hat{H} = \frac{\hbar\omega}{2}\left(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right)\,,$$

when written in terms of the ladder operators $\hat{a}$ and $\hat{a}^\dagger$ which satisfy

$$[\hat{a}, \hat{a}^\dagger] = 1\,.$$

7.1 Consider a normalised wave function $\psi(t, x)$ which satisfies

$$\hat{a}\,\psi(t, x) = \alpha e^{-i\omega t}\psi(t, x)\,,$$

for some real constant α. Compute the expectation values $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$ when the system is described by $\psi(t, x)$ above, where

$$\hat{x} = \sqrt{\frac{\hbar}{2\omega}}(\hat{a}^\dagger + \hat{a})\,, \quad \hat{p} = i\sqrt{\frac{\hbar\omega}{2}}(\hat{a}^\dagger - \hat{a})\,.$$

7.2 For this same wave function, also compute the energy expectation value $\langle\hat{H}\rangle$.

7.3 Connect the wave function $\psi(t, x)$ to what you know about the classical harmonic oscillator. What does this particular wave function describe?

7.4 Compute $(\Delta\hat{x})^2(\Delta\hat{p})^2$ at $t = 0$. Comment on your result.

Q8 Consider a unit-size box containing two non-interacting particles of unit mass $m = 1$. The particles have coordinates x_1 and x_2 respectively, so

$$0 \leq x_1 \leq 1\,, \quad 0 \leq x_2 \leq 1\,.$$

Use units for which $\hbar = 1$.

8.1 Write down the expansion of a generic wave function for the two particles in the box, on a basis of normalised energy eigen-wavefunctions for this system. Motivate your answer.

8.2 Assume that at a particular time, the system is in the state described by the wave function

$$\psi(x_1, x_2, t = t_0) = C\left[\sin(2\pi x_1)\sin(2\pi x_2) + \frac{1}{2}\sin(3\pi x_1)\sin(\pi x_2)\right]\,,$$

for some normalisation constant C. Compute C.

8.3 Determine the probability density $P(x_1)$ for the system described by this wave function.

8.4 Compute the expectation value $\langle x_1 \rangle$ at $t = t_0$.

8.5 At $t = t_0$, the position of particle 2 is measured, and found to be $x_2 = 1/4$. Describe what you now know about the wave function, and give the probability density $P(x_1)$ just after this measurement.

8.6 Does the measurement of the position of particle 2 change the expectation value $\langle x_1 \rangle$? Motivate your answer.