Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH2581-WE01

Title:
Algebra II

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 (a) Find all irreducible polynomials in $(\mathbb{Z}/2)[x]$ of degree less than or equal to two. Carefully explain your reasoning.

(b) Factorise the polynomial $x^5 + x^4 + x^2 + \bar{1}$ into irreducible polynomials in the ring $(\mathbb{Z}/2)[x]$. Justify your answer.

Q2 (a) Let R be an integral domain with finitely many elements. Show that R is a field.

(b) Let S be a commutative ring with finitely many elements. Show that every prime ideal of S is also maximal.

(c) Give an example of a commutative ring with 9 elements which is not an integral domain. Justify your answer.

(d) Give an example of a commutative ring with 9 elements which is an integral domain. Justify your answer.

Q3 **3.1** Let G be a finite group. Show that for all $g \in G$ the order of g, $\operatorname{ord}(g)$, divides the order $|G|$ of G. (Carefully state all results that you use.)

3.2 Let G be a finite group of even order $2m$, and let H be a subgroup of order m, where $m \in \mathbb{N}$. Show that H is a normal subgroup of G.

3.3 Let $\phi : D_4 \to A_4$ be a homomorphism. Show that ϕ is not injective.

Q4 **4.1** Determine all *abelian* groups of order 500, up to isomorphism. (Carefully state all results you use.)

4.2 Determine all *proper* subgroups of a group of order 50, up to isomorphism. (Carefully state all results you use.)

SECTION B

Q5 Let R be the set of polynomials in $\mathbb{Q}[x]$ whose constant term is an integer. That is $R = \{a + xh(x) \ : \ a \in \mathbb{Z},\ h(x) \in \mathbb{Q}[x]\}$.

(a) Show that R is a subring of $\mathbb{Q}[x]$, and determine its units.

(b) Show that the irreducible elements in R are $\pm p$ where p is a prime in $\mathbb{Z}$ and the polynomials $f(x)$ that are irreducible in $\mathbb{Q}[x]$ and have constant term ± 1.

(c) Show that x cannot be written as a product of irreducible elements in R.

Q6 (a) State the Chinese Remainder Theorem for P.I.D.s. (You do not need to define the notion of coprime elements.)

(b) Consider the following polynomials in $(\mathbb{Z}/3)[x]$.

$$f_1(x) = x + \bar{1},\ \ f_2(x) = x^3 + \bar{2}x + \bar{1},\ \ f_3(x) = x^2 + x + \bar{2}.$$

(i) Show that the polynomials $f_1(x), f_2(x), f_3(x)$ are pairwise coprime in the ring $(\mathbb{Z}/3)[x]$.

(ii) Find a $g(x) \in (\mathbb{Z}/3)[x]$ that satisfies the following three properties:
a) $g(\bar{2}) = \bar{1}$,
b) $f_2(x)$ divides $g(x) - (x^2 + \bar{2}x + \bar{2})$ in $(\mathbb{Z}/3)[x]$ and
c) $f_3(x)$ divides $g(x) - \bar{2}$ in $(\mathbb{Z}/3)[x]$.

(Hint: Rewrite the above conditions in terms of the image of $g(x)$ in the quotient rings $(\mathbb{Z}/3)[x]/(f_i(x))$ for $i = 1, 2, 3$ and use C.R.T.)

(iii) Determine all polynomials $g(x) \in (\mathbb{Z}/3)[x]$ with the above three properties. What is the smallest possible degree of such a polynomial?

Q7 Let G be a group. Define a map $\alpha : G \times G \to G$ such that for each pair $g, h \in G$,

$$\alpha(g,h) = ghg^{-1}h^{-1}$$

7.1 Show that for any $g, h \in G$, $\alpha(h,g) = \alpha(g,h)^{-1}$.

7.2 Given $g \in G$, let $\Gamma(g) := \{h \in G : \alpha(g,h) = e\}$, where e is the identity element of G. Show that $\Gamma(g)$ is a subgroup of G for each $g \in G$.

7.3 Let $S \subseteq G$ denote the set $S := \{\alpha(g,h) : g, h \in G\}$. Let H be the group generated by S, i.e., the set

$$H := \{\alpha(g_1,h_1)\alpha(g_2,h_2)\cdots\alpha(g_k,h_k) : g_1,\ldots,g_k,h_1,\ldots,h_k \in G, k \geq 1\}$$

of all finite products of elements of S. (You may assume without proof that it is indeed a group.) Show that H is a normal subgroup of G.

7.4 With the notation of the previous part, show that G is an abelian group if, and only if, $H = \{e\}$.

7.5 Determine the corresponding group H when $G = D_6$.

Q8 **8.1** Determine the size of the conjugacy class of $\sigma = (2\,3)(4\,1\,2)(6\,5\,1)$ in S_6.

8.2 Prove or disprove: the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 2 & 1 & 6 & 5 \end{pmatrix}$$

belongs to A_6.

8.3 Determine, with justification, the normal subgroups of S_4. (Carefully state all results that you use.)

8.4 Show that there are *no* elements $\sigma \in S_4$ that commute with all other elements of S_4. (*Hint:* Use the previous part.)