Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH2647-WE01

Title:
Probability II

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 Let $([0,1], \mathcal{B}[0,1], \mathsf{P})$ be the canonical probability space. Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ are two collections of events in $\mathcal{B}[0,1]$, where

$$\mathcal{D}_1 = \{[a,b] : 0 \le a < b < 1\}, \qquad \mathcal{D}_2 = \{[c,d) : 0 \le c < d \le 1\}.$$

(a) Show that every event in $\mathcal{D}_2$ can be written as a countable limit of events in $\mathcal{D}_1$.

(b) Show that every event in $\mathcal{D}_1$ can be written as a countable limit of events in $\mathcal{D}_2$.

(c) Let $\mathcal{G}$ be a sigma-field satisfying $\mathcal{D}_1 \subseteq \mathcal{G} \subseteq \mathcal{B}[0,1]$. Is it true that $\mathcal{D}_2 \subseteq \mathcal{G}$? Justify your answer.

(d) Suppose that a sigma-field $\mathcal{F}$ satisfies $\mathcal{D}_2 \subseteq \mathcal{F} \subseteq \mathcal{B}[0,1]$. Is it true that $\mathcal{D}_1 \subseteq \mathcal{F}$? Justify your answer.

Q2 Let $(X_n)_{n\ge1}$ be random variables such that $X_n \sim \mathsf{Exp}(n^2)$, i.e., $\mathsf{P}(X_n > y) = \mathrm{e}^{-n^2 y}$ for all $y \ge 0$. Define $Y = \sum_{k\ge1} X_k$.

(a) Derive a formula for the expectation $\mathsf{E}Y$.

(b) Is it true that $Y > 0$ is a finite random variable, i.e., $\mathsf{P}(Y < \infty) = 1$? Justify your answer.

SECTION B

Q3 Let b_n be the probability that n independent Bernoulli trials (with individual success probability $p \in (0,1)$) result in an odd number of successes.

(a) Compute b_0, b_1, and b_2.

(b) Use generating functions to derive a simple closed formula for b_n and check that it gives correct values for $n = 0, 1, 2$.

(c) Find $\lim_{n\to\infty} b_n$ and explain your result.

Q4 Let $(Z_k)_{k\geq 1}$ be independent random variables with common distribution $\mathcal{N}(0,1)$. For integer $n \geq 1$, let

$$S_n = \frac{1}{n}\big((Z_1)^2 + (Z_2)^2 + \cdots + (Z_n)^2\big).$$

(a) Show that S_n converges in L^2 and find its limit.

(b) Does S_n converge almost surely? If so, find its limit. If not, explain why.

(c) Does S_n converge in probability? If so, find its limit. If not, explain why.

In your answer you should define the relevant modes of convergence, provide a detailed justification, and give a clear statement of any result you use.