Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH2657-WE01

Title:

Special Relativity and Electromagnetism II

Time:	2 hours		
Additional Material provided:			
Materials Permitted:			
Calculators Permitted:		Models Permitted:	

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 **1.1** Grogu activates a distress beacon that emits radio waves. Mando receives the distress signal while he is aboard his spaceship travelling directly away from Grogu with a relative speed v. Mando turns his spaceship around and now travels directly towards Grogu with the same relative speed v. Calculate v, as a fraction of the speed of light c, given that Mando measures that the frequency of the signal has doubled on turning his spaceship around.

1.2 Inside a nuclear reactor a pair of neutrons move off in opposite directions, with a relative speed $7c/8$. An engineer measures the speed of one of the neutrons to be $c/2$. Calculate the speed that the engineer measures for the other neutron, as a fraction of the speed of light c.

Q2 **2.1** A ball of radius a contains a charge density in its interior given by $\rho = qr^5/a^8$, where r is the distance to the centre of the ball and q is a constant. The surface of the ball is covered in a thin material with a constant surface charge density σ, to ensure that the covered ball has zero total charge. Calculate σ in terms of q and a.

2.2 In a magnetostatics problem, in a region of space the magnetic vector potential is

$$\mathbf{A} = (\lambda xy^2, \lambda yz^2, \lambda zx^2),$$

where λ is a constant. Calculate the magnetic field $\mathbf{B}$ and the current density $\mathbf{J}$, in this region of space, in terms of λ and the spatial coordinates x, y, z.

SECTION B

Q3 **3.1** Derive the condition on the transformation $x'^{\mu} = L^{\mu}{}_{\nu} x^{\nu}$ for it to preserve the interval $\eta_{\mu\nu} x^{\mu} x^{\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric.

Use this result to prove that the product of two Lorentz transformations is also a Lorentz transformation.

3.2 Two particles, one of which is initially at rest, have the same rest mass and collide elastically. After the collision the pair of outgoing particles have the same speed and the angle between their directions of motion is θ.

Let p^{μ} denote the initial 4-momentum of the particle at rest and q^{μ} the initial 4-momentum of the other particle, which takes the form

$$q^{\mu} = \gamma m(c, \mathbf{v}), \qquad \text{where} \qquad \gamma = \frac{1}{\sqrt{1 - |\mathbf{v}|^2/c^2}}$$

and $\mathbf{v}$ is a constant non-zero vector, with c the speed of light.

Let $\tilde{p}^{\mu}$ and $\tilde{q}^{\mu}$ denote the 4-momenta of the particles after the collision, with

$$\tilde{q}^{\mu} = \tilde{\gamma} m(c, \tilde{\mathbf{v}}), \qquad \text{where} \qquad \tilde{\gamma} = \frac{1}{\sqrt{1 - |\tilde{\mathbf{v}}|^2/c^2}}$$

and $\tilde{\mathbf{v}}$ is another constant vector.

(a) Show that $q^{\mu} q_{\mu}$ is independent of $\mathbf{v}$.

(b) Use the conservation of energy to obtain a formula for $|\tilde{\mathbf{v}}|$ in terms of γ.

(c) Calculate $(p^{\mu} + q^{\mu})(p_{\mu} + q_{\mu})$ in terms of m, c and γ.

(d) Calculate $(\tilde{p}^{\mu} + \tilde{q}^{\mu})(\tilde{p}_{\mu} + \tilde{q}_{\mu})$ in terms of $m, c, |\tilde{\mathbf{v}}|$ and θ.

(e) Obtain a formula for γ in terms of $c, |\tilde{\mathbf{v}}|$ and θ.

(f) Find θ, given that initially the kinetic energy of one of the particles is equal to four times its rest energy.

Q4 **4.1** Electric charge is contained within a ball of radius R that is centred at the origin. The electric charge density vanishes outside the ball but inside it is given by

$$\rho = \frac{qR^3}{(R^3 + r^3)^2},$$

where q is a positive constant and $r = |\mathbf{r}|$ is the distance to the origin.

Calculate the electric field at position $\mathbf{r}$ due to the ball, in terms of q, R, ε_0, and sketch its magnitude as a function of r.

4.2 A wire forms a square of side length $2L$ and carries a current I.

Use the Biot-Savart law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3},$$

to calculate the magnitude of the magnetic field at the centre of the square, in terms of I, μ_0, L.

You may use without proof the result that

$$\int_0^L \frac{du}{(u^2 + L^2)^{3/2}} = \frac{1}{L^2\sqrt{2}}.$$