Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH2707-WE01

Title:
Markov Chains II

Time:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 Let $I = \{1, 2, 3, 4, 5\}$. Let P be the following matrix, indexed by I:

$$P = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & p_{33} & \frac{1}{2} & 0 \\ p_{41} & 0 & \frac{1}{2} & 0 & p_{45} \\ p_{51} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Suppose that P is stochastic and that the state 2 does *not* lead to the state 5, i.e., $p_{25}^{(n)} = 0$ for all $n \geq 0$.

(a) What must the values of p_{33}, p_{41}, p_{45}, and p_{51} be?

(b) What are the associated communicating classes?

(c) Which of the associated communicating classes are recurrent?

(d) Explicitly describe the set of stationary distributions of P.

For each problem you should show all your workings and justify your calculations with suitable explanations.

Q2 Consider a king on a 5×5 chessboard that starts at the square labelled 'A' in Figure 1.

2.1 Suppose the king moves by choosing its next location uniformly at random from all adjacent squares. Prove from first principles that the first time the king lands on a black square is a stopping time.

2.2 Suppose the king moves by choosing its next location uniformly at random from all adjacent white squares. What is the expected number of moves for the king to get back to where it starts?

For each problem you should show all your workings and justify your calculations with suitable explanations.

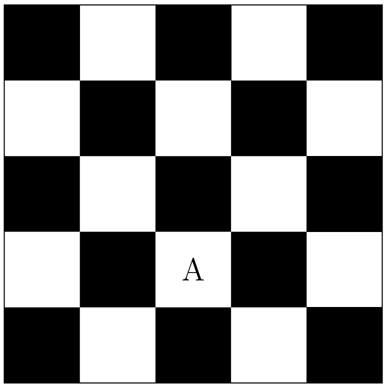

Figure 1: A 5×5 chessboard. The king moves between squares. It is allowed to move from its current square to any of the adjacent squares (squares that touch it). From the square labelled 'A' there are eight adjacent squares, four of which are white.

SECTION B

Q3 (a) Let $I = \{0, 1, 2, \ldots, N\}$, and let P be the transition matrix with entries $p_{i,i+1} = p_{i,i-1} = \frac{1}{2}$ for $i = 1, 2, \ldots, N-1$, and $p_{0,0} = p_{N,N} = 1$. If $(X_n)_{n\geq 0}$ is Markov(δ_i, P) for some $i \in I$, what is the probability that the Markov chain reaches the state N before the state 0?

Hint: remember the solution to the linear recurrence $x_i = \frac{1}{2}x_{i-1} + \frac{1}{2}x_{i+1}$ has the general form $x_i = \alpha + \beta i$ for some constants α, β.

(b) Consider simple random walk on the graph in Figure 2 with initial state the state labelled 'x'. You may assume $N \geq 2$. What is the probability that the Markov chain reaches one of the states B or C before the state A?

Hint: it may help to think of the central state as being labelled N.

For each problem you should show all your workings and justify your calculations with suitable explanations.

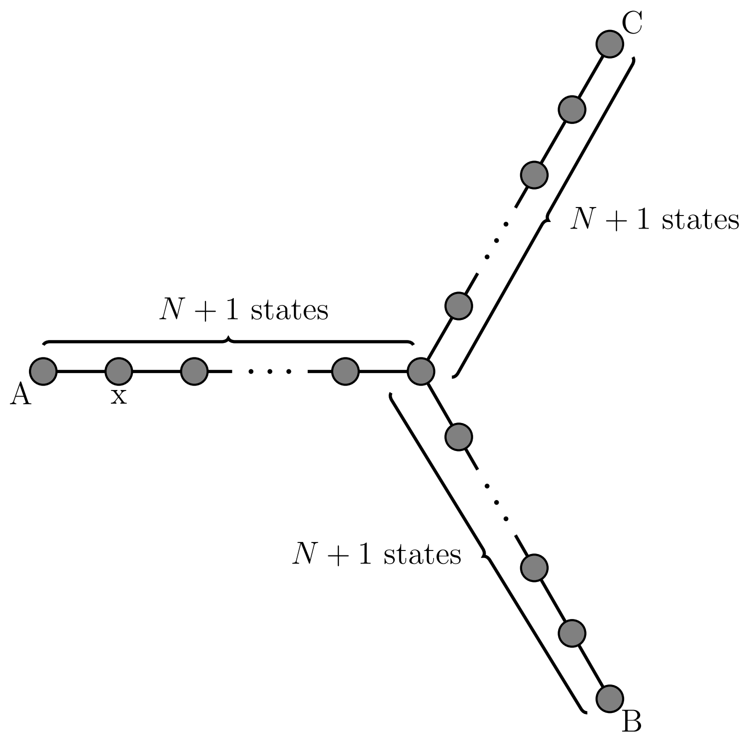

Figure 2: The graph for Q3 (b). The figure indicates that from 'A' to the central state there are $N+1$ states in total, including the state labelled 'A' and the central state. There are also $N+1$ states from 'B' and from 'C' to the central state.

Q4 Let $I = \mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, and let $P = Q + R$ be the transition matrix formed by the sum of the matrices $Q = (q_{ij})_{i,j\in\mathbb{Z}}$ and $R = (r_{ij})_{i,j\in\mathbb{Z}}$ with entries

$$q_{ij} = \begin{cases} \frac{2}{5} & j = i+1 \\ \frac{2}{5} & j = i-1 \\ 0 & \text{else.} \end{cases}, \qquad r_{ij} = \begin{cases} \frac{1}{5} & j = 0 \\ 0 & \text{else} \end{cases}.$$

Suppose $(X_n)_{n\geq 0}$ is a Markov chain with transition matrix P.

(a) Is $(X_n)_{n\geq 0}$ transient or recurrent?

(b) Does there exist an invariant measure for P?

(c) Does there exist an invariant distribution for P?

For each problem you should show all your workings and justify your calculations with suitable explanations.