Durham University

# EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
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| May/June | 2023 | MATH3011-WE01 |

| Title: |
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| Analysis III |

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| Time: | 3 hours | |
| Additional Material provided: | | |
| Materials Permitted: | | |
| Calculators Permitted: | No | Models Permitted: Use of electronic calculators is forbidden. |

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| Instructions to Candidates: | Answer all questions.<br>Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.<br>Students must use the mathematics specific answer book. |

| Revision: | |
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ED01/2023

## SECTION A

**Q1** (a) (i) Let $\phi$ be a simple measurable function taking only nonnegative values. State the definition of the integral $\int \phi$ of $\phi$.

(ii) Let $f$ be a measurable function taking values in the nonnegative extended reals. State the definition of the integral $\int f$ of $f$.

(b) State the *Monotone Convergence Theorem.*

(c) Let $f, g$ be measurable functions taking values in the nonnegative extended reals. Show that if $f \leq g$ then $\int f \leq \int g$.
(Hint: you may use that for nonnegative simple measurable functions $\phi \leq \psi$ implies $\int \phi \leq \int \psi$.)

**Q2** In this question all functions are assumed to have domain $[0, 1]$ and take values in the extended reals. Let $g$ be a measurable and integrable function and for each $t \in [0, 1]$ consider the function

$$h_t(x) = \begin{cases} g(x) & \text{if } 0 \leq x < t, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Is $h_t$ integrable? Justify your answer.

(b) State the *Dominated Convergence Theorem.*

(c) Define $F(t) = \int h_t$. Show that if $(t_n)_n$ is any sequence in $[0, 1]$ that converges to $t$ as $n \to \infty$, then $F(t_n)$ converges to $F(t)$ as $n \to \infty$.
(Hint: use linearity of the integral for integrable functions.)

(d) State what it means for a real-valued function to be continuous at a point $x \in [0, 1]$.

**Q3** **3.1** Let $E \subset \mathbb{R}$ be measurable and let $1 \leq p < \infty$.

(a) State the definition of $L^p(E)$.

(b) Prove that if $f \in L^p(E)$ and $E_\alpha = \{x \in E : |f(x)| \geq \alpha\}$, $\alpha > 0$, then $\mu(E_\alpha) \to 0$ as $\alpha \to \infty$ (where $\mu$ denotes the Lebesgue measure).

**3.2** Let $q > 1$. If $f \in L^q[0, 1]$, then is it true that $f \in L^r[0, 1]$ for $1 \leq r < q$? Give a full justification of your response.

**Q4** Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let the norm derived from $\langle \cdot, \cdot \rangle$ be denoted by $\| \cdot \|$.

(a) State Bessel's Inequality for an orthonormal set $U = \{u_\alpha : \alpha \in I\} \subset \mathcal{H}$.

(b) Suppose that for any $x \in \mathcal{H}$, $x = \sum_{\alpha \in I} \langle x, u_\alpha \rangle u_\alpha$ where the sum has at most countably many non-zero terms and converges unconditionally in $\mathcal{H}$. Prove Parseval's Identity.

## SECTION B

**Q5** Let $C \subset [0,1]$ be the middle-third Cantor set.

(a) Explain why $C$ is a Borel set.

(b) Show that $C$ has Lebesgue measure 0.

(c) State what it means for a point $x$ to be an accumulation point of a set $E \subset \mathbb{R}$.

(d) Define the function $g : [0,1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} x & \text{if } x \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Is $g$ Riemann integrable? Justify your answer.

**Q6** In the following $A, B, B_n, E, E_n \subset \mathbb{R}$. Recall that the symmetric difference of two sets $A$ and $B$ is

$$A \,\Delta\, B = (A - B) \cup (B - A).$$

(a) Show that outer measure is monotonic: namely if $A \subset B$ then $\mu^*(A) \leq \mu^*(B)$.

(b) Show that if $\mu^*(E \,\Delta\, B_n) \to 0$ as $n \to \infty$ then $\mu^*(B_n) \to \mu^*(E)$ as $n \to \infty$.

(c) Suppose that $E_n$ is a finite union of intervals and suppose that $\mu^*(E \,\Delta\, E_n) \to 0$ as $n \to \infty$. Show that we have

$$\mu^*(A) = \mu^*(E \cap A) + \mu^*(E^c \cap A).$$

**Q7** Let $E \subseteq \mathbb{R}$ be measurable. Let $1 \leq p, q < \infty$. Consider the linear space $L^p(E) \times L^q(E)$ where addition and scalar multiplication are defined pointwise as $(f_1, g_1) + (f_2, g_2) = (f_1 + f_2, g_1 + g_2)$ and $\lambda(f, g) = (\lambda f, \lambda g)$ for $\lambda \in \mathbb{C}$. Define $\|(\cdot,\cdot)\| : L^p(E) \times L^q(E) \to \mathbb{R}_{\geq 0}$ as

$$\|(f, g)\| = (\|f\|_{L^p}^2 + \|g\|_{L^q}^2)^{1/2}, \quad (f, g) \in L^p(E) \times L^q(E).$$

**7.1** Prove that $\|(\cdot,\cdot)\|$ is a norm on $L^p(E) \times L^q(E)$ for $1 \leq p, q < \infty$.

**7.2** Prove that $(L^p(E) \times L^q(E), \|(\cdot,\cdot)\|)$ is a Banach space for $1 \leq p, q < \infty$.

**7.3** Using your response to **7.2** or otherwise, state conditions that normed linear spaces $X, Y$, with norms $\|\cdot\|_X$, $\|\cdot\|_Y$ respectively, must satisfy to ensure that $X \times Y$ with norm $\|(x, y)\| = (\|x\|_X^2 + \|y\|_Y^2)^{1/2}$ is a Banach space.

**7.4** Give an example of normed linear spaces $(X, \|\cdot\|_1)$, $(Y, \|\cdot\|_2)$, such that $X \times Y$ with norm $\|(x, y)\| = (\|x\|_1^2 + \|y\|_2^2)^{1/2}$ is not a Banach space. Briefly justify your response.

**Q8** **8.1** Consider $(L^3[0,1], \|\cdot\|_{L^3})$. Is the norm $\|\cdot\|_{L^3}$ derived from an inner product? Justify your response.

**8.2** Let $E \subset \mathbb{R}$ be measurable. Let $f \in L^1(E)$. Prove that

$$\lim_{n\to\infty} \int_E f(x)(\cos(nx))^2 = \frac{1}{2}\int_E f(x).$$

You may use without proof that since $f \in L^1(E)$, for any $\epsilon > 0$, there exists a step function $\psi$ such that $\|f - \psi\|_{L^1} < \epsilon$. You should otherwise give a full justification.