Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH3021-WE01

Title:
Differential Geometry III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 Let $\alpha : \mathbb{R} \to \mathbb{R}^3$, $\alpha(t) = (t^2, t^3, t^4)$. Let $K \subset \mathbb{R}$ be the set of all those $t \in \mathbb{R}$ for which the following holds : *α is not regular at t and α has vanishing curvature at t.* Give the set K. Give the domain of the torsion of the curve α. Calculate the torsion of the curve α.

Q2 Let $c : [0, 2\pi] \to \mathbb{R}^2$, $c(u) = (u - \sin(u), 1 - \cos(u))$.

(a) Calculate the torsion and curvature of c in terms of $\sin(u/2)$. Identify the domains of curvature and torsion. You may use (without proof) the trigonometric relation
$$\cos u = \cos^2(u/2) - \sin^2(u/2).$$

(b) Prove or disprove that the length $L(c)$ of the curve c satisfies $L(c) < 4$.

Q3 Let $S \subset \mathbb{R}^3$ be a regular surface, $\lambda > 0$, and
$$\widetilde{S} := \{\lambda \mathbf{p} : \mathbf{p} \in S\},$$
with the corresponding map $\Phi : S \to \widetilde{S}$, $\Phi(\mathbf{p}) = \lambda \mathbf{p}$. Prove that $\widetilde{S}$ is regular. Prove that Φ is a smooth map of regular surfaces which maps geodesics to geodesics. Derive relations between the Gauss curvature of S at $\mathbf{p} \in S$ and the Gauss curvature of $\widetilde{S}$ at $\Phi(\mathbf{p}) \in \widetilde{S}$.

Q4 Let $S \subset \mathbb{R}^3$ be a regular surface and $P \subset \mathbb{R}^3$ be a Euclidean plane such that the intersection $S \cap P$ is the trace of a smooth regular curve $c : I \to S$, with $I \subset \mathbb{R}$ a suitable interval. Show the following: If c is constant speed and if the planes $T_{c(t)}S$ and P intersect perpendicularly for all $t \in I$, then c is a geodesic.

SECTION B

Q5 Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(a) Let $\beta : [a, b] \to \mathbb{R}^2$ be a smooth regular parametrized plane curve (not necessarily unit speed). Define the curvature of β. Show that the curvature of β is given by
$$\kappa_\beta(u) = \frac{(\beta'(u)J) \cdot \beta''(u)}{\|\beta'(u)\|^3},$$
where we consider $\beta'(u), \beta''(u)$ as row vectors and where $\mathbf{a} \cdot \mathbf{b}$ denotes the Euclidean inner product of the row vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$.

(b) Let $\alpha : [0, L] \to \mathbb{R}^2$ be a smooth unit speed plane curve with strictly positive curvature $\kappa_\alpha > 0$. Define an involute β of α and calculate the curvature of the involute $\beta : [0, L] \to \mathbb{R}^2$ which passes through the point $\alpha(0)$.

Q6 Let $S \subset \mathbb{R}^3$ be the set of all points $(x, y, z) \in \mathbb{R}^3$ satisfying $x \sin z = y \cos z$.

(a) Show that S is a regular surface by showing that it is the pre-image of a regular value.

(b) Show that $\mathbf{x}(u, v) = (v\cos u, v\sin u, u)$ is a parametrisation of S which satisfies the following property : *For all* $p \in S$ *there exists* $(u, v) \in \mathbb{R}^2$ *with* $\mathbf{x}(u, v) = p$. Using this parametrisation, calculate the mean curvature and the Gauss curvature of S. Prove or disprove that one of these curvatures at any point $(x, y, z) \in S$ depends only on the value $x^2 - y^2$.

Q7 Let $I \subset \mathbb{R}$ be an interval, $M > 0$, and $\alpha : I \to \mathbb{R}^3$ be injective and a smooth unit speed curve with curvature $\kappa : I \to (0, M) \subset \mathbb{R}$. Let $S_r \subset \mathbb{R}^3$ be parametrized by

$$\mathbf{x}(u, v) = \alpha(u) + r(\cos(v) \cdot \mathbf{n}(u) + \sin(v) \cdot \mathbf{b}(u)).$$

(a) Prove that for sufficiently small $r > 0$, $\mathbf{x}(u, v)$ is regular.

(b) Show that, for sufficiently small $r > 0$, the area of the surface S_r is given by $\text{area}(S_r) = 2\pi r L(\alpha)$, where $L(\alpha)$ is the length of α.

(c) Assume that α is a simple closed curve and $r > 0$ small enough so that $\mathbf{x}(u, v)$ is regular. Let $K : S_r \to \mathbb{R}$ be the Gauss curvature of S_r. Explain why we then have

$$\int_{S_r} K dA = 0.$$

Q8 Let $S \subset \mathbb{R}^3$ be the surface of revolution obtained by rotating a simple smooth regular curve $\alpha(u) = (f(u), 0, g(u))$ with $f > 0$ about the z-axis of $\mathbb{R}^3$. Let $c : (a, b) \to S$ be a unit speed geodesic, $r(s) > 0$ be the distance from the point $c(s) = (c_1(s), c_2(s), c_3(s))$ to the z-axis, and $\theta(s)$ be the angle at $c(s)$ between the circle $S \cap \{z = c_3(s)\}$ and the unit vector $c'(s)$.

(a) Let $e_3 = (0, 0, 1)$ and $s \in (a, b)$ be fixed. Writing $c(s) = (r(s)\cos\beta, r(s)\sin\beta, z)$ for suitable $\beta = \beta(s)$, and parametrising the parallel through $c(s)$ by

$$\gamma(t) = (r(s)\cos(\beta + t), r(s)\sin(\beta + t), z),$$

show that $\gamma'(0) = -c(s) \times e_3$. Conclude from this that

$$-(c(s) \times e_3) \cdot c'(s) = r(s)\cos\theta(s).$$

(b) Let $\mathbf{x}$ be the standard surface parametrisation associated to the curve α. Calculate $\mathbf{x}_u \times \mathbf{x}_v$ and show that any vector perpendicular to $T_{\mathbf{p}}S$ must lie in the plane spanned by $\mathbf{p} \in S$ and e_3.

(c) Show by differentiation that there is a constant $C \in \mathbb{R}$ such that

$$(c(s) \times c'(s)) \cdot e_3 = C$$

for all $s \in (a, b)$. You may use (b) and the fact that c is a geodesic.

(d) Use the above to prove or disprove that $r(s)\cos\theta(s) = C$ for all $s \in (a, b)$.