EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH3031-WE01

Title:
Number Theory III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 Let $\alpha \in \mathbb{C}$ be an algebraic integer.

(a) Define the set of conjugates of α.

(b) Assume that $\alpha \neq 0$. Show that some conjugate of α has absolute value at least 1. Carefully justify your reasoning. (Hint: Consider $N_{\mathbb{Q}(\alpha)}(\alpha)$.)

Q2 (a) Let $K = \mathbb{Q}(\sqrt{7})$. Compute the discriminant $\Delta_K(2, 3+\sqrt{7})$.

(b) Let $K = \mathbb{Q}(\theta)$, where θ is a root of the polynomial $7x^2 - 3$. Determine the ring of integers $\mathcal{O}_K$ and give a generator for its group of units $\mathcal{O}_K^\times$.

Q3 (a) Let K be a number field and I and J two ideals of $\mathcal{O}_K$ such that $I \subseteq J$. Recall that every ideal of $\mathcal{O}_K$ is a full lattice in K and hence has a discriminant. Prove that if $\Delta_K(I) = \Delta_K(J)$, then $I = J$.

(b) Find all the ideals of $\mathbb{Z}[\sqrt{-21}]$ that contain the element 10 and have norm 10.

Q4 Let $K = \mathbb{Q}(\sqrt{-5})$ and consider the following ideals of $\mathcal{O}_K$:

$$\begin{aligned}\mathfrak{p} &= (2, 1+\sqrt{-5}),\\ \mathfrak{q} &= (3, 1+\sqrt{-5}),\\ \mathfrak{r} &= (3, 1-\sqrt{-5}).\end{aligned}$$

(a) Using Kummer–Dedekind or otherwise, show that $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ are prime ideals and find their norms.

(b) Show that $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ are not principal ideals.

(c) Show that in the class group $Cl(\mathcal{O}_K)$, we have the relations

$$[\mathfrak{p}]^2 = e, \qquad [\mathfrak{p}][\mathfrak{q}] = e.$$

SECTION B

Q5 (a) Let R be any integral domain and suppose that $x \in R$ has a factorisation

$$x = up_1 \cdots p_n,$$

where u is a unit and the p_i are *prime elements*. Show that the above factorisation is the unique factorisation of x into a product of *irreducible* elements, up to ordering of the factors and multiplication by units. (Hint: Let $x = vq_1 \cdots q_m$ be a factorisation where v is a unit and the q_i are irreducible.)

(b) Let r be an element of an integral domain R and consider the principal ideal (r) of R. Show that if (r) is a prime ideal, then r is a prime element.

(c) It is known that the ring $R = \mathbb{Z}[\sqrt{-5}]$ is not a UFD (i.e., does not have unique factorisation into irreducibles). Show that, nevertheless, the element $5 \in R$ does factor uniquely into a product of irreducibles (up to units and ordering of factors).

Q6 Let $\theta \in \mathbb{C}$ be a root of the polynomial $x^3 - 3$ and let $K = \mathbb{Q}(\theta)$.

(a) Compute $\Delta_K(\mathbb{Z}[\theta])$.

(b) Show that $\mathcal{O}_K = \mathbb{Z}[\theta]$.

Q7 Let $K = \mathbb{Q}(\sqrt{d})$, where $d \in \mathbb{Z}$ is squarefree and $d \not\equiv 1 \pmod 4$. Let $a, b, c \in \mathbb{Z}$ with a and c non-zero and such that

$$c \mid a, \quad c \mid b \quad \text{and} \quad ac \mid (b^2 - dc^2).$$

(a) Show that there exist $n, r \in \mathbb{Z}$ such that $bn - dc = ar$.

(b) Prove that

$$\{a, b + c\sqrt{d}\}$$

is a generating basis (i.e., a $\mathbb{Z}$-basis) for the ideal $(a, b + c\sqrt{d})$ of $\mathcal{O}_K$.

(c) Compute the norm of the ideal $(6, 5 - \sqrt{7})$.

Q8 Determine the class group of $K = \mathbb{Q}(\sqrt{-57})$. You may use the Minkowski bound, given by $B_K = \left(\dfrac{4}{\pi}\right)^t \dfrac{n!}{n^n}\sqrt{|\Delta_K|}$.