Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH3041-WE01

Title:
Galois Theory III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 (a) Determine the minimal polynomial of $\sqrt{3-2\sqrt{2}}$ over $\mathbb{Q}$.

(b) Find all complex roots of the polynomial $x^4+6x^2-\sqrt{10}x+3/2$.

Q2 (a) Determine the Galois group of the polynomial $f(x)=x^3-x+1$ over $\mathbb{Q}$.

(b) Determine the Galois group of the polynomial $f(x)=x^3-x+1$ over $\mathbb{Q}(\sqrt{-23})$.

Q3 (a) Suppose that a non-zero polynomial $f(x)\in\mathbb{F}_p[x]$ has a repeated root in some field extension of $\mathbb{F}_p$. Prove that $f(x)$ is reducible in $\mathbb{F}_p[x]$.

(b) Let α be a root of $g(x)=x^4+x^3+1\in\mathbb{F}_2[x]$ in some field extension. Determine the degree of the extension $\mathbb{F}_2(\alpha)/\mathbb{F}_2$ and find the minimal polynomial of α^{-1}.

Q4 (a) Find a normal closure N of the extension $\mathbb{Q}(\sqrt[4]{7})/\mathbb{Q}$.

(b) Describe the structure of the Galois group of $N/\mathbb{Q}$ from the previous part by specifying generators and relations.

SECTION B

Q5 Let $L=\mathbb{R}(x_1,x_2,x_3)$ be the field of rational expressions in independent indeterminates x_1,x_2,x_3 with real coefficients, and set

$$e_1=x_1+x_2+x_3,\quad e_2=x_1x_2+x_1x_3+x_2x_3,\quad e_3=x_1x_2x_3.$$

(a) Find an expression for $x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2$ as a polynomial in e_1,e_2,e_3.

(b) Show that L is a Galois extension of $K=\mathbb{R}(e_1,e_2,e_3)$ and determine the structure of $\mathrm{Gal}(L/K)$.

(c) Let $\alpha=x_1+2x_2+3x_3\in L$. By considering the action of the Galois group or otherwise, show that $L=K(\alpha)$.

Q6 (a) Find the minimal polynomial of $\theta=\sqrt{5-\sqrt{5}}$ over $\mathbb{Q}$ and show that $K=\mathbb{Q}(\theta)$ is a Galois extension of $\mathbb{Q}$.

(b) Determine the structure of $\mathrm{Gal}(K/\mathbb{Q})$ and describe how its elements act on θ.

(c) Find all fields M where $\mathbb{Q}\subset M\subset K$.

Q7 Let $L=\mathbb{Q}(\zeta_{25})$ where ζ_{25} is a primitive 25^{th} root of unity.

(a) Show that L contains unique subfields M and K such that $[L:M]=5$ and $[L:K]=4$.

(b) Explain why L/M is a cyclic extension and find $\alpha\in M$ such that $L=M(\sqrt[5]{\alpha})$.

(c) Explain why L/K is a cyclic extension and show that there is no $\beta\in K$ such that $L=K(\sqrt[4]{\beta})$.

Q8 (a) Show that the polynomial $f(x)=x^3+2x+1$ is irreducible in $\mathbb{F}_3[x]$. Given a root θ of $f(x)$ in a splitting field, express the other two roots of $f(x)$ as linear combinations of the basis $1,\theta,\theta^2$ over $\mathbb{F}_3$.

(b) By using $f(x)=x^3+2x+1$ or otherwise, construct an irreducible polynomial in $\mathbb{F}_3[x]$ which has degree 39.

(c) Given prime p and integer $n\geq 1$, prove that $x^{p^n}-x\in\mathbb{F}_p[x]$ is the product of all irreducible monic polynomials in $\mathbb{F}_p[x]$ with degree dividing n.