



EXAMINATION PAPER

Examination Session: May/June	Year: 2023	Exam Code: MATH30620-WE01
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Title: Topology V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	<p>Answer all questions.</p> <p>Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks.</p> <p>Students must use the mathematics specific answer book.</p>	
		Revision:

SECTION A

Q1 Let $X = \{1, 2, 3, 4, 5\}$, and consider the collections of subsets

$$\mathcal{A} = \{\{1, 2, 3, 4\}, \{2, 3, 4, 5\}\} \text{ and } \mathcal{B} = \{\{1, 2, 3, 4\}, \{3\}, \{4\}, \{5\}\}.$$

- (a) Find $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$, the smallest topologies containing each collection. Is \mathcal{A} a basis for $\tau_{\mathcal{A}}$? Is \mathcal{B} a basis for $\tau_{\mathcal{B}}$? Explain briefly.
- (b) For each topology, determine:
 whether it is compact or not;
 whether it is Hausdorff or not;
 whether it is connected or not.

Q2 Let d be any metric on a non-empty set X . Consider the functions:

$$d_{\min} : X \times X \rightarrow \mathbb{R} \text{ given by } d_{\min}(x, y) = \min\{1, d(x, y)\}$$

$$d_{\max} : X \times X \rightarrow \mathbb{R} \text{ given by } d_{\max}(x, y) = \max\{1, d(x, y)\}$$

- (a) Show that the triangle inequality holds for both these functions, but that only one of them is a metric.
- (b) For $x \in X$ and $r \in [0, \infty)$, the open ball $B(x; r)$ is defined by

$$B(x; r) = \{y \in X \mid d(x, y) < r\}.$$

We can define corresponding open balls $B_{\min}(x; r)$, $B_{\max}(x; r)$, for d_{\min} , d_{\max} respectively. In terms of the $B(x; r)$, describe the $B_{\min}(x; r)$ and $B_{\max}(x; r)$ for different values of r .

- Q3** (a) State what it means for two topological spaces to be homotopy equivalent.
- (b) Let X , Y and Z be topological spaces and suppose that X and Y are homotopy equivalent. Show that $X \times Z$ is homotopy equivalent to $Y \times Z$.

- Q4** (a) In terms of a suitable graph, state the definition of the Euler characteristic of a closed surface. Briefly justify why the definition is independent of your choice of (suitable) graph.
- (b) If closed surfaces S_1 and S_2 are homeomorphic, prove that their Euler characteristics agree; that is, $\chi(S_1) = \chi(S_2)$.

SECTION B

- Q5** (a) Prove that a compact subset A of a Hausdorff space X is closed in X .
 (b) State the Heine-Borel theorem about compactness in \mathbb{R}^n with the standard topology. Prove **one** direction of this result.
 (c) Let

$$S = \{f(t) \mid t \in [1, \infty)\} \subseteq \mathbb{R}^2 \text{ where } f(t) = \left(1 + \frac{1}{t}\right) (\cos(2\pi t), \sin(2\pi t)).$$

Which points in $\mathbb{R}^2 \setminus S$ are limit points of S ? Show that these are limit points; you are not asked to show that these are the only ones.

- (d) Show in **two** ways that S is not compact: i) using Heine-Borel;
 ii) directly, using the definition of compactness. [You may use visual or geometric arguments, but for full marks define any sets explicitly.]

Q6 Let

$$C = \{(\cos \theta, \sin \theta, t) \mid \theta \in [-\pi, \pi], t \in [-1, +1]\} \subseteq \mathbb{R}^3 \text{ (with standard topology),}$$

and define the following two equivalence relations on C :

$$(x_1, y_1, z_1) \sim_1 (x_2, y_2, z_2) \iff \begin{cases} (x_1, y_1, z_1) = (x_2, y_2, z_2) \text{ OR} \\ x_1 = x_2, y_1 = y_2, \{z_1, z_2\} = \{-1, +1\} \end{cases}$$

$$(x_1, y_1, z_1) \sim_2 (x_2, y_2, z_2) \iff \begin{cases} (x_1, y_1, z_1) = (x_2, y_2, z_2) \text{ OR} \\ z_1 = z_2 = -1 \text{ OR } z_1 = z_2 = +1 \end{cases}$$

- (a) The quotient spaces C/\sim_1 and C/\sim_2 are homeomorphic to two well-known surfaces S_1 and S_2 . *Without proof* say which these are.
 (b) Consider a function $f : C \rightarrow \mathbb{R}^3$ of the form

$$f(\cos \theta, \sin \theta, t) = (k(t) \cos \theta, k(t) \sin \theta, t).$$

For a suitable function $k : [-1, +1] \rightarrow \mathbb{R}$, the image of f is homeomorphic to one of the surfaces S_1 or S_2 from part (a). Decide which, and find such a function k .

From now on we write this surface and the corresponding equivalence as S and \sim .

- (c) For certain subsets A and B of C , the space C/\sim could also be defined as $(C/A)/B$. Find these subsets. Why could we not write C/\sim as $C/(A \cup B)$? Describe $C/(A \cup B)$ in words, or a picture.
 (d) As usual, we define the maps

$$\pi : C \rightarrow C/\sim \text{ given by } \pi(x, y, z) = [(x, y, z)],$$

$$\bar{f} : C/\sim \rightarrow S \text{ given by } \bar{f}[(x, y, z)] = f(x, y, z),$$

so that $f = \bar{f} \circ \pi$. Show that \bar{f} is both well-defined and injective.

- (e) Show that $\bar{f} : C/\sim \rightarrow S$ is a homeomorphism.
 You may use the following result from lectures: If X is compact, Y is Hausdorff, and $f : X \rightarrow Y$ is a continuous bijection, then f is a homeomorphism.
 If you use any other results then refer to them clearly.

- Q7** (a) State the definitions of (i) a finite simplicial complex and (ii) a triangulation of a topological space.
- (b) Let K and L be 2-dimensional, finite simplicial complexes and suppose that $f : K \rightarrow L$ is a simplicial map such that, for some fixed $r \in \mathbb{N}$, every simplex in L is the homeomorphic image of exactly r disjoint simplices in K . We say that K is an r -fold covering of L . Find a relationship between r and the Euler characteristics of K and L .
- (c) Suppose that the sphere S^2 is triangulated by a finite simplicial complex K , where K is an r -fold covering of a simplicial complex L . Determine all possible values of $r \in \mathbb{N}$ and, if L also triangulates a closed surface M , determine all possibilities for M up to homeomorphism.

Q8 Let X be a topological space and let $x_0 \in X$.

- (a) State the definition of the fundamental group $\pi_1(X, x_0)$ of the based space (X, x_0) , including the definition of the group operation.
- (b) If X is a path-connected space with non-trivial fundamental group $\pi_1(X, x_0)$, show that the identity map $\text{id}_X : X \rightarrow X$ is not null homotopic.
- (c) Suppose now that the topology on X is induced by a metric $d_X : X \times X \rightarrow [0, \infty)$ and let ΩX be the set whose elements are continuous maps $f : [0, 1] \rightarrow X$ with $f(0) = f(1) = x_0$, which is equipped with a metric

$$D : \Omega X \times \Omega X \rightarrow [0, \infty)$$

$$(f, g) \mapsto \sup\{d_X(f(t), g(t)) \mid t \in [0, 1]\}.$$

If the fundamental group $\pi_1(X, x_0)$ is trivial, prove that ΩX is path connected with respect to the topology induced by the metric D , being careful to ensure that any maps you introduce are continuous.