EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH3071-WE01

Title:
Decision Theory III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX83 series or FX85 series.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 A certain company must decide whether to market a particular product (decision d_1) or not (decision d_2). If decision d_1 is taken, then there is considered to be probability 0.5 that sales will be good (event G) and probability 0.5 that sales will be bad (event B). Given good sales, expected profits are 40 million pounds, while, given bad sales, expected losses are 20 million pounds. Choosing d_2 gives no profit or loss.

Before making a decision, the company has the option of commissioning some market research. The research will either make a positive recommendation (event P) or a negative recommendation (event N). The research is judged to be 75% reliable, i.e.

$$P(P|G) = P(N|B) = 0.75$$

(a) Suppose that the market research makes a negative recommendation. Find the conditional probability of good sales.

(b) Solve this decision problem, finding the optimal decision procedure and the expected profit, assuming that the market research is free and that the company wishes to maximise expected monetary returns.

(c) Find and interpret

(i) the expected monetary value of the market research;

(ii) the risk profile for the decision procedure chosen in part (b);

(iii) the expected value of perfect information, assuming that the initial market research is free.

Q2 In a certain decision problem, suppose that the reward for the decision involves two attributes X and Y. Denote the reward in which $X = x, Y = y$ as (x, y).

Suppose that you have two decision choices. Decision d_1 gives reward $(4, 4)$ with certainty. Decision d_2 gives reward $(6, 6)$ with probability 0.5 and reward $(0, 0)$ with probability 0.5.

Suppose that you judge the two attributes to be mutually utility independent. Suppose that you fix U(0,0) = 0, and that you judge each marginal utility to be linear, namely $U(x, 0) = x$ and $U(0, y) = y$.

Suppose also that (i) you are indifferent between reward $(2, 0)$ and reward $(0, 4)$ and (ii) you are indifferent between reward $(4, 2)$ and reward $(2, 5)$.

(a) Evaluate your utility function for rewards as a function of x and y. (Any properties of utility that you require should be stated clearly but need not be proved.)

(b) Interpret what the utility function reveals about your combined attitude towards the two attributes.

(c) Which of the two decisions d_1 and d_2 would you prefer?

Q3 (a) Consider the decision table with utilities given below, where you have the choice between four actions $(a_1, \ldots, a_4)$ and there are four unknown states $(\theta_1, \ldots, \theta_4)$. Uncertainty about the real state is reflected by the probabilities $P(\theta_1) = 0.4$ and $P(\theta_i) = 0.2$ for $i \in \{2, 3, 4\}$. Determine the optimal decisions according to each of the following optimality criteria: (1) Maximum expected utility; (2) Maximisation of minimum utility; (3) Minimisation of maximum regret.

	θ_1	θ_2	θ_3	θ_4
a_1	7	1	8	3
a_2	2	6	1	3
a_3	4	4	6	2
a_4	7	5	2	5

(b) Consider the pay-off table below, where person R has the options R1 to R4 and person C has the options C1 to C4. The entry (x, y) denotes pay-off x to R and y to C when the corresponding options are played. Determine all Pure Nash Equilibria for the game with this pay-off table.

	C1	C2	C3	C4
R1	(6,6)	(0,7)	(5,3)	(1,6)
R2	(9,1)	(6,4)	(8,2)	(4,4)
R3	(3,8)	(5,5)	(1,4)	(7,3)
R4	(5,5)	(3,2)	(7,6)	(8,9)

Q4 Consider the following procedure to combine preference rankings of $m \geq 3$ people over $k \geq 3$ options:

Each person $i \in \{1, \ldots, m\}$ assigns scores $s_i(\cdot)$ to the options, which in case of strict preferences only are integers from 1 to k, with 1 for the least preferred option and k for the most preferred option. In case of indifference among two or more options, those options get the corresponding average score. Next, for each option x with set of individual scores $\{s_i(x), i = 1, \ldots, m\}$, one minimal score and one maximal score are deleted and the group score $S(x)$ is defined to be the sum of the remaining $m-2$ scores. Finally, the group preference ordering is defined based on these group scores in the logical manner, with a higher score reflecting a higher group preference, so e.g. $x >_g y$ if and only if $S(x) > S(y)$ and $x \sim_g y$ if and only if $S(x) = S(y)$.

(a) Apply this procedure to the preference orderings in the table below, to derive the group preference ordering. For simplicity of notation, $<$ and $\sim$ are used to denote an individual's preference or indifference between two options.

Person	Preference ordering
1	$a < b < c < d < e$
2	$c < a < d < b \sim e$
3	$b \sim d \sim e < c < a$
4	$e < b < a \sim d < c$

(b) For each of the axioms in Arrow's Impossibility Theorem, explain in detail whether or not it is satisfied by this general procedure. For each axiom which is not satisfied, illustrate this via an example.

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(c) Explain briefly the main role of the Pareto Condition in Arrow's theory of social choice.

SECTION B

Q5 (a) Explain what is meant by a utility function on a set of rewards. Discuss the relevance of utility functions to problems of decision making.

(b) Suppose that an individual prefers to receive 400 pounds, for sure, than to receive a gamble paying one thousand pounds with probability 0.5 and paying nothing with probability 0.5. Suppose the individual also prefers a gamble paying one thousand pounds with probability 0.25 and nothing with probability 0.75 to a gamble paying four hundred pounds with probability 0.5 and nothing with probability 0.5.

(i) Show that there is no utility function on amounts of money which is consistent with the above preferences.

(ii) Identify which of the assumptions on preferences that are required to ensure the existence of a utility function is broken by the above preferences.

(c) Explain what is meant by the certainty equivalent and the risk premium for a gamble with monetary outcomes.

Consider the following gamble. A fair coin is tossed repeatedly until the first time that a head is tossed. If the coin has been tossed n times, then the reward is 2^n pounds.

Suppose that an individual has utility for x pounds of form $U(x) = \sqrt{x}$. Find the certainty equivalent and risk premium for this gamble.

Using this example, discuss the role of utility in problems with financial payoffs.

Q6 (a) Suppose that the random quantity θ has a Gamma distribution, parameters $\alpha, \beta > 0$. Show that, for any integer k for which $\alpha + k > 0$,

$$\begin{aligned} E_\theta(\theta^k) &= \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\beta^k}, \text{ if k} > 0 \\ &= \frac{1}{(\alpha+k)(\alpha+k+1)\cdots(\alpha-1)\beta^k}, \text{ if k} < 0 \end{aligned}$$

[The Gamma distribution, parameters $\alpha, \beta > 0$ has, for positive x, probability density function given by $p(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$.]

(b) We wish to estimate the parameter, $\theta > 0$, of an exponential distribution.
[The exponential distribution has density $f(x) = \theta e^{-\theta x} \qquad (x \geq 0)$.]
We wish to produce an estimate d, for θ, with loss function

$$L(\theta, d) = \frac{(\theta - d)^2}{\theta^3}.$$

The prior distribution for θ is a Gamma distribution, parameters $\alpha > 3$ and $\beta > 0$.
Find the Bayes decision, and show that the Bayes risk is

$$\frac{\beta}{(\alpha-1)(\alpha-2)}$$

(c) Suppose that we take a sample of size n of independent observations, each exponentially distributed with parameter θ. Suppose that we observe $\underline{x} = (x_1, \ldots, x_n)$.
Show that the posterior distribution for θ, given $\underline{x}$, is also a Gamma distribution with parameters $\alpha + n, \beta + \sum_{i=1}^n x_i$.

(d) Find the Bayes decision and Bayes risk, given $\underline{x}$.

(e) Show that the Bayes risk of the sampling procedure for a sample of size n is

$$\frac{\beta}{(\alpha-1)(\alpha+n-2)}$$

(f) Find the optimal choice of sample size, if each sample costs c.

Q7 Two people, U and V, agree to solve a joint decision problem via bargaining. They have five options, A, B, C, D, E, for which their individual utilities are as follows

	A	B	C	D	E
U	3	1	4	5	6
V	6	4	5	3	2

They decide that, if they fail to reach agreement, they will settle for a status quo option, for which they both have utility 0.

(a) Sketch the feasible region and identify the Pareto boundary for this problem.

(b) Find the Nash Point for this problem and specify the bargain corresponding to the Nash Point.

(c) Derive the Equitable Distribution Point for this problem and specify the corresponding bargain.

(d) Suppose that a new option, F, becomes available, for which U has utility 2 and V has utility 7. Without further detailed calculations to derive the Nash Point and the Equitable Distribution Point for this case, explain whether or not the inclusion of this new option would change the Nash Point or the Equitable Distribution Point. For any change, explain for U and for V whether or not they are better off with the new option F included than without it.

Q8 Consider the following pay-off table for a two-person zero-sum game, where R chooses $R1$ or $R2$, and C chooses $C1, C2, C3, C4$ or $C5$. The pay-offs to R are as follows

	$C1$	$C2$	$C3$	$C4$	$C5$
$R1$	4	2	6	3	5
$R2$	6	8	2	4	3

The pay-off to C is minus the pay-off to R.

(a) Identify all dominated strategies.

(b) Find the minimax strategies for R and C and the value of this game.

(c) Replace the value 3 for the combination of strategies $(R2, C5)$ in the table above by x. Specify all solutions to this game, i.e. the minimax strategies and value of the game, as function of x, for all $x \in [0, 3]$.

(d) Explain briefly if it is easier, harder or similarly difficult for a player to determine an optimal strategy in such a game if they aim to maximise their expected utility rather than use the minimax criterion.