EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH3091-WE01

Title:
Dynamical Systems III

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 A dynamical system is governed by the equations

$$\dot{x} = y + 2x(9 - x^2 - y^2)(1 - (x^2 + y^2)^2)$$
$$\dot{y} = -x + 2y(9 - x^2 - y^2)(1 - (x^2 + y^2)^2)$$

Rewrite these equations in polar coordinates (r, φ) and show that the equations for the φ and r are decoupled. Sketch the phase flow for this system in polar and Cartesian coordinates. Please label all interesting points and subsets in this flow and explain your reasoning.

Q2 A linear system is governed by the equation

$$\ddot{x} - 4\dot{x} + 68x = 0$$

(a) Write this system in first order form by introducing $y = 1/13(3x - \dot{x})$.

(b) Explicitly find a similarity transformation matrix M, which is used to write the first order linear system from part (a) in the diagonal form. Write the original system in the new basis and find its solution.

(c) Apply the similarity transformation to obtain solution in the original basis.

(d) Sketch the phase flow for this linear system in the original coordinate system.

Q3 Consider the one-dimensional dynamical system $\dot{x} = f(x, \mu)$, and suppose f depends smoothly on x and a parameter μ.

(a) Give the necessary conditions for $f(x, \mu)$ to have a local bifurcation at a point $x_\star$.

(b) Now suppose that $f(x, \mu) = (x^2 - 2)^2 - \mu^2$. Solve the conditions that you found in (a) to find the locations of the four bifurcation points.

(c) What are the types of bifurcations at these four points? State without proof which ones are robust under small perturbations and which ones are not.

Q4 A general planar dynamical system is given by

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y)$$

(a) (i) Suppose that the only critical points are $(x, y) = (0, 0)$ and $(x, y) = (2, 0)$. Suppose also that the linearised systems near these critical points take the form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x - 2 \\ y \end{pmatrix}$$

Let γ_R be a circle around the origin with radius R. Give the Poincaré indices $I(\gamma_{1/7})$ and $I(\gamma_{17/7})$.

(ii) Could either of $\gamma_{1/7}$ or $\gamma_{17/7}$ be a periodic orbit? Explain your answer.

(b) Consider a different system with

$$f(x, y) = x^3 - xy^2 - 8x$$
$$g(x, y) = \frac{4}{3}y^3 + x^5 + 6y + \frac{2}{3}$$

Prove that no periodic orbit lies entirely in the disk $x^2 + y^2 < 1/3$.

SECTION B

Q5 A two-dimensional dynamical system is governed by the equations

$$\begin{aligned} \dot{x} &= -2x - 3y^2\,, \\ \dot{y} &= -3x^2 + 2y\,. \end{aligned} \tag{1}$$

(a) Work out all fixed points of this system and determine their nature.

(b) Using the insight from the part (a) sketch the phase flow for this system.

(c) Explain what is the Hamiltonian system and show that the system (1) is Hamiltonian. Integrate equations and find the Hamiltonian for this system. You can set the integration constant to zero.

(d) Use the Hamiltonian that you have found in part (c), to work out the equation of the path through the point $(0, 0)$. State the Stable Manifold theorem and explain how this theorem works in this example.

Q6 A dynamical system is governed given by the equations

$$\begin{aligned} \dot{x} &= -y - x^2 \\ \dot{y} &= -x + y^2 \end{aligned}$$

Answer the following questions:

(a) Work out all critical points of this system.

(b) By linearizing the system, work out the nature of fixed points and sketch the local behaviour near fixed points.

(c) Show that for the particular value(s) of the parameter α, paths $y = \alpha x$ solve this system. Find the solution for this path in the parametric form $(x(t), y(t))$, and work out the asymptotic behaviour of this path. Say what is special about this path.

(d) Sketch the full phase diagram based on the previous analysis and label all the special paths you have found.

Q7 Two dynamical systems, defined by the equations $\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y})$ and $\dot{\mathbf{Y}} = \mathbf{G}(\mathbf{Y})$ have critical points at $\mathbf{y} = \mathbf{0}$ and $\mathbf{Y} = \mathbf{0}$, respectively.

(a) Define what is meant by the statement that the two dynamical systems are topologically conjugate.

(b) In a particular case, the dynamical systems are linear systems $\dot{\mathbf{y}} = A'\mathbf{y}$ and $\dot{\mathbf{Y}} = B'\mathbf{Y}$, where

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B' = \begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}$$

By finding the general solutions to these equations and considering the homeomorphism h defined by the equations $Y_1 = y_1,\ Y_2 = (y_2)^5$, show that the two systems are topologically conjugate.

(c) A further linear system is given by the equations $\dot{\mathbf{x}} = A\mathbf{x}$ where the matrix A is given as

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

Show that there exists a linear transformation M such that if $\mathbf{x} = M\mathbf{y}$ then $\mathbf{y}$ satisfies the equations given for $\mathbf{y}$ given in part (b).

(d) The solution of the system in part (c) is (you don't have to show this)

$$\phi(t, \mathbf{x_0}) = \begin{pmatrix} (a+b)e^t - be^{-t} \\ be^{-t} \end{pmatrix}$$

with a, b constants. By combining the linear transformations M of part (c) with the homeomorphism h given in part (b), find a homeomorphism between $\mathbf{x}$ and $\mathbf{Y}$ and use this together with the explicit solutions to show that the dynamical systems $\dot{\mathbf{x}} = A\mathbf{x}$ and $\dot{\mathbf{Y}} = B'\mathbf{Y}$ are topologically conjugate.

Q8 Let $K(x, y) = \frac{1}{2}y^2 + \frac{2}{3}x^3 - \frac{3}{2}x^2 + x$ and consider the dynamical system

$$\dot{x} = \partial_y K = y, \qquad \dot{y} = -\partial_x K = -2x^2 + 3x - 1$$

(a) Without computing the critical points, give an argument why none of them can be asymptotically stable. Now find the critical points and determine their types.

(b) Sketch the phase portrait, showing every type of orbit. Hint: there is a homoclinic orbit through one of the critical points.

(c) Define what is meant by an ω-limit point.

(d) Identify every ω-limit set of the above system.