EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH30920-WE01

Title: Mathematical Biology V

Time:	3 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

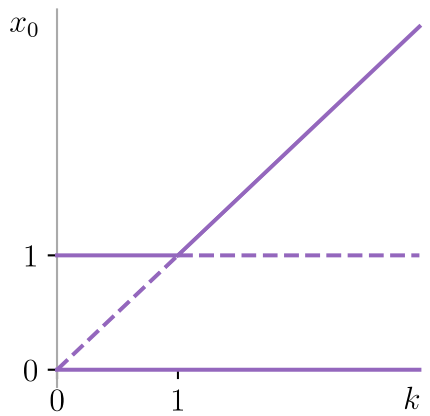
Q1 Single-species population models

1.1 Consider the following models for the growth of a single population, $x(t)$, over time, t. For each model, find any equilibria and determine their stability. Further, state one good and one bad feature of each model in terms of its biological interpretation, and say how you might change each model to address the bad feature you have mentioned.

(i) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = x$,

(ii) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = -(x-a)(x-b)$, where a and b are positive constants with $a < b$.

1.2 Consider the following bifurcation diagram, which shows how the equilibria x_0 of a single-species population model depend on a single parameter, k:



(i) Write down all the equilibria, x_0, shown on the graph.

(ii) Hence write down a model for a population $x(t)$ in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, k)$$

which would produce this bifurcation diagram.

Q2 Two-species interaction The growth of two interacting species, $x(t)$ and $y(t)$, over time, t, is modelled by the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(4 - x^2 - y),$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(-1 + x).$$

2.1 Find the permissible equilibria of the system and classify them.

2.2 Draw the phase plane of y against x for the region $x, y \geq 0$, including sample trajectories, clearly marking nullclines and equilibria.

Q3 What could have made all these cobwebs...? Let u_n and v_n be populations of flies and spiders at some discrete time. We model their interaction within one generation by

$$u_n = \alpha u_{n-1} - u_{n-1}v_{n-1},$$
$$v_n = u_{n-1}v_{n-1} - \beta v_{n-1},$$

where α, and β are positive parameters.

3.1 Find all equilibria, and determine any conditions on the parameters for each equilibrium to be permissible (besides $\alpha > 0$ and $\beta > 0$).

3.2 Determine conditions for each equilibrium to be stable.

3.3 Briefly say how this stability analysis compares to the standard Lotka–Volterra system seen in lectures,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = au - uv,$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = uv - bv,$$

where $(u_0, v_0) = (0, 0)$ is always permissible and unstable, and $(u_0, v_0) = (1/b, 1/a)$ is always permissible but is a centre (so not asymptotically stable).

Q4 Don't be a square (PDE stability) For this question, consider the rectangular domain $\Omega = [0, L_x] \times [0, L_y]$.

4.1 Consider a single population u in Ω evolving according to

$$\frac{\partial u}{\partial t} = D\nabla^2 u + ru\left(1 - \frac{u}{K}\right), \tag{1}$$

where D, r and K are positive constants, and where $u = 0$ on the boundary of Ω (homogeneous Dirichlet conditions). What do the terms in this model represent, and what is the biological interpretation of the boundary conditions?

4.2 Find the homogeneous equilibria of (1), remembering to check the boundary conditions.

4.3 State the eigenvalues ρ of the problem

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\rho w,$$

with $w = 0$ on the boundary of Ω (homogeneous Dirichlet conditions). That is, write down all values of ρ (you do not need to compute these).

4.4 For each homogeneous equilibrium of (1) found in question **4.2**, compute a condition on the parameters which determines its linear stability.

SECTION B

Q5 Fourier 'tr-ants-forms' A population of ants live along a kitchen worktop in a student house. While one student attempts to remove them with ant killer, another decides to model the ant population size with a partial differential equation. Modelling the worktop as an infinitely-long line, the student's model for the population over space and time, $u(x,t)$, is

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - \alpha u, \quad u(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ A & \text{if } 0 \le x \le L \\ 0 & \text{if } x > L \end{cases}, \quad u(\pm\infty, t) = 0,$$

where D, α, A and L are positive constants and $x \in \mathbb{R}$.

5.1 Interpret the terms in this equation, including the boundary and initial conditions, in the context of the ants and their environment. By considering your interpretation (i.e. not by solving the system), what long-term behaviour do you expect from this model?

5.2 By writing

$$u = U\widehat{u}, \quad x = X\widehat{x}, \quad t = T\widehat{t},$$

where variables with hats are dimensionless, show that the model can be written in dimensionless form as, once hats are removed,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \beta u, \quad u(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x > 1 \end{cases}, \quad u(\pm\infty, t) = 0, \qquad (2)$$

for some $\beta \ge 0$. You should specify the values of U, X and T you chose, as well as the value of β, in terms of the original problem's parameters.

5.3 Using Fourier transforms, solve the fundamental problem for (2) and then solve the full problem. You may find the following useful:

$$\mathcal{F}(k)[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} f(x)\,\mathrm{d}x,$$

$$\mathcal{F}^{-1}(x)[g(k)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} g(k)\,\mathrm{d}k,$$

$$\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2+bx+c}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}\mathrm{e}^{c}\mathrm{e}^{b^2/(4a)},$$

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-t^2}\,\mathrm{d}t.$$

Confirm that your prediction for the long-term behaviour of the model ($t \to \infty$) in question **5.1** was correct.

5.4 Very briefly explain why this approach would not work if the partial differential equation in (2) were

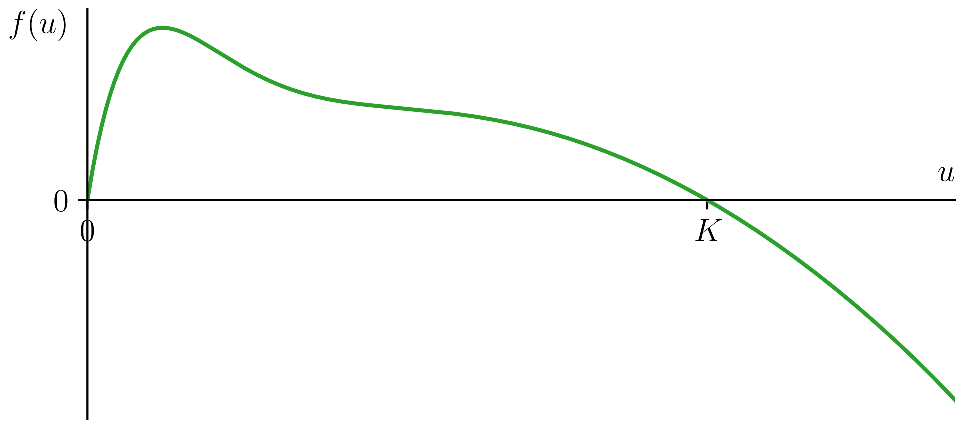
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \beta u^2,$$

with the same boundary and initial conditions.

Q6 Travelling waves in coral reefs Consider a model for the growth of coral in one spatial dimension,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u), \tag{3}$$

where $u(x,t)$ is the density of coral at position x and time t, and $f(u)$ is given by the diagram below:



6.1 By looking at the diagram, state the stability of the equilibria of the homogeneous form of (3).

6.2 How is (3) similar to the Fisher–Kolmogorov equation? How is it different?

6.3 We are now going to look for a travelling wave solution, $u(z)$, where $z = x - ct$ and $c > 0$ is the wavespeed. Given your answer to question **6.1**, propose boundary conditions for $u(z)$ at $z = \pm\infty$.

6.4 Now write (3) for $u(z)$. Perform a linear stability analysis on the homogeneous equilibria to find that the speed of the wave, given that waves travel at their lowest possible speed, is

$$c = \alpha\sqrt{f'(\beta)},$$

where you should state the value of α and β.

6.5 Briefly say what properties of f you used to show that travelling waves exist. That is, what did the graph of $f(u)$ have to look like to find travelling waves?

Q7 Massive chemotaxis Consider a version of the Keller–Segel model of cell chemotaxis given by:

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u - \boldsymbol{\nabla} \cdot (\chi(u,v)\boldsymbol{\nabla} v), \tag{4}$$

$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + uf(v) - g(v), \tag{5}$$

where $u(t, \boldsymbol{x})$ is the population of cells, $v(t, \boldsymbol{x})$ the chemoattractant, D_u, D_v are strictly positive parameters, and χ, f, and g are strictly positive functions. Consider this model posed on a generic fixed bounded spatial domain Ω. Assume that both u and v satisfy Neumann boundary conditions on the boundary. That is, for $\boldsymbol{x} \in \partial\Omega$, we have $\boldsymbol{n} \cdot \boldsymbol{\nabla} u = \boldsymbol{n} \cdot \boldsymbol{\nabla} v = 0$, where $\boldsymbol{n}$ is the unit normal vector.

7.1 Give a biological interpretation to all of the parameters and functions in this model.

7.2 Show that the boundary conditions and the form of equation (4) together imply that the total mass of the cells, given by

$$M(t) = \int_\Omega u(t, \boldsymbol{x})\, \mathrm{d}\boldsymbol{x},$$

does not change in time. You will need to make use of the divergence theorem, which states that

$$\int_\Omega \boldsymbol{\nabla} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{v}\, \mathrm{d}S,$$

where $\boldsymbol{v}$ is a vector field, $\partial\Omega$ is the domain boundary, $\boldsymbol{n}$ the outward unit normal, and $\mathrm{d}S$ a surface element of the domain.

7.3 Find all homogeneous equilibria of this system (u_0, v_0). Explain why assuming that $g(v)/f(v)$ is a strictly increasing smooth function of v is enough to guarantee a unique value of v_0 given a value of u_0.

7.4 Assume you know the solutions to the spatial eigenvalue problem,

$$\nabla^2 w = -\rho w,$$

where $w(\boldsymbol{x})$ satisfies Neumann boundary conditions on $\partial\Omega$. Perform a linear stability analysis of the system (4)-(5) around a spatially homogeneous equilibrium, (u_0, v_0), finding a quadratic equation for an instability growth rate λ as a function of a given spatial eigenvalue ρ. You do not need to solve this equation for λ.

7.5 Show that, in the absence of any transport terms ($D_u = D_v = \chi = 0$), we must have that $g'(v_0) > u_0 f(u_0)$ to prevent instabilities. Describe why $\lambda = 0$ is a solution in the spatially homogeneous case, but that it does not indicate a growing instability.

Q8 Swift–Hohenberg meets Dirichlet Consider the Swift–Hohenberg equation on an interval $[0, \pi]$:

$$\frac{\partial u}{\partial t} = ru - \left(k_c^2 + \frac{\partial^2}{\partial x^2}\right)^2 u - u^3, \tag{6}$$

where $r, k_c \in \mathbb{R}$ are parameters.

8.1 Write down the homogeneous form of the equation. Determine all homogeneous equilibria of this equation and their stability. Classify any bifurcations that occur in the homogeneous system.

8.2 Consider the problem

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -\rho w, \quad w(0) = w(\pi) = 0.$$

State the eigenfunctions $w(x)$ and eigenvalues ρ (you do not need to compute these).

8.3 (i) Which homogeneous boundary conditions for the problem (6) allow the use of the eigenfunctions $w(x)$ found above to perform a linear stability analysis?

(ii) Which of the homogeneous equilibria of (6) found in question **8.1** satisfy these boundary conditions?

(iii) Perform a linear stability analysis of the Swift–Hohenberg equation (6) around any equilibria satisfying these boundary conditions. Describe precise conditions that guarantee a pattern-forming instability on this finite spatial domain.