

FORMULA SHEET for MATH 3101/41820 : FLUID MECHANICS

Some vector identities:

$$\nabla \cdot (f\mathbf{A}) = (\nabla f) \cdot \mathbf{A} + f\nabla \cdot \mathbf{A} \quad (1)$$

$$\nabla \times (f\mathbf{A}) = (\nabla f) \times \mathbf{A} + f\nabla \times \mathbf{A} \quad (2)$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \Delta \mathbf{A} \quad (3)$$

$$\mathbf{A} \cdot \nabla \mathbf{A} = \frac{1}{2} \nabla |\mathbf{A}|^2 - \mathbf{A} \times (\nabla \times \mathbf{A}) \quad (4)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \quad (5)$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \mathbf{A} - \mathbf{A} \cdot \nabla \mathbf{B} + \mathbf{A} \nabla \cdot \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{A} \quad (6)$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \quad (7)$$

In cylindrical coordinates (r, θ, z) :

$$\nabla f = \text{grad } f = \mathbf{e}_r \partial_r f + \frac{\mathbf{e}_\theta}{r} \partial_\theta f + \mathbf{e}_z \partial_z f \quad (8)$$

$$\nabla \cdot \mathbf{A} = \text{div } \mathbf{A} = \frac{1}{r} \partial_r (r A_r) + \frac{1}{r} \partial_\theta A_\theta + \partial_z A_z \quad (9)$$

$$\nabla \times \mathbf{A} = \text{rot } \mathbf{A} = \left(\frac{1}{r} \partial_\theta A_z - \partial_z A_\theta \right) \mathbf{e}_r + (\partial_z A_r - \partial_r A_z) \mathbf{e}_\theta + \frac{1}{r} (\partial_r (r A_\theta) - \partial_\theta A_r) \mathbf{e}_z \quad (10)$$

$$\Delta f = \frac{1}{r} \partial_r (r \partial_r f) + \frac{1}{r^2} \partial_{\theta\theta} f + \partial_{zz} f \quad (11)$$

$$\Delta \mathbf{A} = \left(\Delta A_r - \frac{1}{r^2} A_r - \frac{2}{r^2} \partial_\theta A_\theta \right) \mathbf{e}_r + \left(\Delta A_\theta + \frac{2}{r^2} \partial_\theta A_r - \frac{1}{r^2} A_\theta \right) \mathbf{e}_\theta + \mathbf{e}_z \Delta A_z \quad (12)$$

$$\mathbf{B} \cdot \nabla \mathbf{A} = \mathbf{e}_r \left(\mathbf{B} \cdot \nabla A_r - \frac{B_\theta A_\theta}{r} \right) + \mathbf{e}_\theta \left(\mathbf{B} \cdot \nabla A_\theta + \frac{B_\theta A_r}{r} \right) + \mathbf{e}_z (\mathbf{B} \cdot \nabla A_z) \quad (13)$$

In spherical coordinates (r, θ, ϕ) :

$$\nabla f = \text{grad } f = \mathbf{e}_r \partial_r f + \frac{\mathbf{e}_\theta}{r} \partial_\theta f + \frac{\mathbf{e}_\phi}{r \sin \theta} \partial_\phi f \quad (14)$$

$$\nabla \cdot \mathbf{A} = \text{div } \mathbf{A} = \frac{1}{r^2} \partial_r (r^2 A_r) + \frac{1}{r \sin \theta} \partial_\theta (A_\theta \sin \theta) + \frac{1}{r \sin \theta} \partial_\phi A_\phi \quad (15)$$

$$\begin{aligned} \nabla \times \mathbf{A} = \text{rot } \mathbf{A} = & \frac{\mathbf{e}_r}{r \sin \theta} (\partial_\theta (A_\phi \sin \theta) - \partial_\phi A_\theta) + \frac{\mathbf{e}_\theta}{r} \left(\frac{1}{\sin \theta} \partial_\phi A_r - \partial_r (r A_\phi) \right) \\ & + \frac{\mathbf{e}_\phi}{r} (\partial_r (r A_\theta) - \partial_\theta A_r) \end{aligned} \quad (16)$$

$$\Delta f = \frac{1}{r} \partial_{rr} (r f) + \frac{1}{r^2 \sin \theta} \partial_\theta (\sin \theta \partial_\theta f) + \frac{1}{r^2 \sin^2 \theta} \partial_{\phi\phi} f \quad (17)$$

$$\begin{aligned} \Delta \mathbf{A} = & \left(\Delta A_r - \frac{2}{r^2} A_r - \frac{2}{r^2 \sin \theta} [\partial_\theta (\sin \theta A_\theta) + \partial_\phi A_\phi] \right) \mathbf{e}_r \\ & + \left(\Delta A_\theta + \frac{2}{r^2} \partial_\theta A_r - \frac{A_\theta}{r^2 \sin^2 \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \partial_\phi A_\phi \right) \mathbf{e}_\theta \\ & + \left(\Delta A_\phi + \frac{2}{r^2 \sin \theta} \partial_\phi A_r + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \partial_\phi A_\theta - \frac{A_\phi}{r^2 \sin^2 \theta} \right) \mathbf{e}_\phi \end{aligned} \quad (18)$$

$$\begin{aligned} \mathbf{B} \cdot \nabla \mathbf{A} = & \mathbf{e}_r \left(\mathbf{B} \cdot \nabla A_r - \frac{B_\theta A_\theta}{r} - \frac{B_\phi A_\phi}{r} \right) + \mathbf{e}_\theta \left(\mathbf{B} \cdot \nabla A_\theta - \frac{B_\phi A_\phi}{r} \cot \theta + \frac{B_\theta A_r}{r} \right) \\ & + \mathbf{e}_\phi \left(\mathbf{B} \cdot \nabla A_\phi + \frac{B_\phi A_r}{r} + \frac{B_\phi A_\theta}{r} \cot \theta \right) \end{aligned} \quad (19)$$

Bessel functions $u(r) = J_n(r)$ and $u(r) = Y_n(r)$ are solutions to the ODE

$$r^2 u'' + r u' + (r^2 - n^2) u = 0. \quad (20)$$

Both $J_n(r)$ and $Y_n(r) \rightarrow 0$ as $r \rightarrow \infty$; $J_n(0) = \delta_{n0}$, and $|Y_n(r)| \rightarrow \infty$ as $r \rightarrow 0$.