Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May/June	2023	MATH3101-WE01

Title:

Fluid Mechanics III

Time:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.

Instructions to Candidates:	Answer all questions. Section A is worth 40% and Section B is worth 60%. Within each section, all questions carry equal marks. Students must use the mathematics specific answer book.

Revision:	

ED01/2023

SECTION A

Q1 Consider the flow $\boldsymbol{u} = \sin(t)x\boldsymbol{e}_x + \sin(t)y\boldsymbol{e}_y + \cos(t)z\boldsymbol{e}_z$.

(a) Find the particle paths.

(b) Is this flow compressible or incompressible?

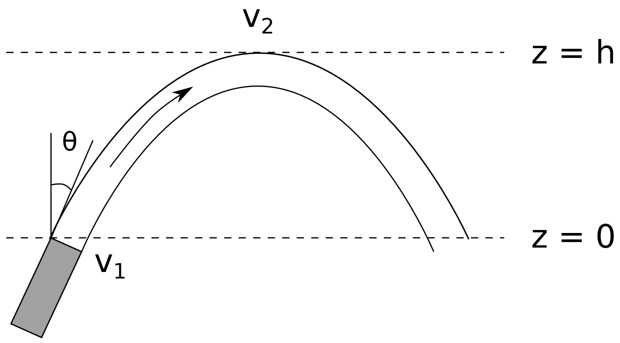
(c) Consider a material volume (D_t) in this flow bounded at $t = 0$ by a cylinder of radius R and height h centred at the origin such that

$$D_0 = \{(x, y, z) : -h/2 \leq z \leq h/2, x^2 + y^2 \leq R^2\}.$$

Find an expression for how the volume of D_t changes over time.

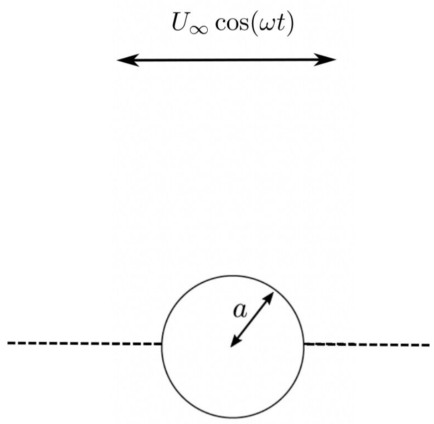
Q2 (a) Write down the equations that describe an ideal, steady, inviscid flow with a body force of the form $\boldsymbol{f} = -\boldsymbol{\nabla} U$ and show that Bernoulli's function is constant along streamlines in such a flow.

(b) Consider the steady laminar flow from a hose at an angle θ to the vertical as shown.



The flow leaves the hose at speed v_1 and is travelling horizontally at speed v_2 when the flow reaches its highest point. The only force acting on the flow is gravity so that $\boldsymbol{f} = -g\boldsymbol{e}_z$. Assuming that pressure is constant within the flow use Bernoulli's function to find h in terms of g, v_1 and θ.

Q3 We want to non-dimensionalise the governing equations for the given oscillatory flow over a cylinder of radius a (see the figure). The flow far from the cylinder is given as $U_\infty \cos(\omega t)$, where ω is the frequency of far field oscillations.



(a) Write down the viscous incompressible 2D Navier Stokes equations (without body force)and then non-dimensionalise them using the following scaling

$$\mathbf{u}' = \frac{1}{U_\infty}\mathbf{u}, \qquad \mathbf{x}' = \frac{1}{a}\mathbf{x}, \qquad t' = \omega t, \qquad p' = \frac{1}{\rho\omega a U_\infty}p.$$

Express the dimensionless equations in terms of

$$Re = \frac{U_\infty a}{\nu} \quad \text{and} \quad \epsilon = \frac{U_\infty / a}{\omega}.$$

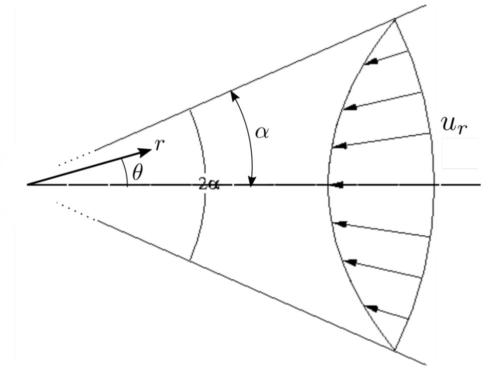
If $\epsilon \ll 1$ and $Re = O(\epsilon)$, which terms balance each other at leading order?

(b) Repeat part (a), but change the pressure scaling to

$$p' = \frac{a}{\mu U_\infty}p.$$

Remember that $\nu = \mu/\rho$. If $\epsilon \ll 1$ and $Re = O(\epsilon^2)$ what are the equations that we retrieve at leading order?

Q4 Consider a converging channel flow with the half-angle α as shown in the figure.



We assume that the flow is 2D ($u_z = 0$ and $\partial/\partial z = 0$), and satisfies the incompressible, steady state Navier Stokes equations in polar coordinates. For the velocity, we assume the following form

$$\boldsymbol{u} = u_r \boldsymbol{e_r} + u_\theta \boldsymbol{e_\theta}, \quad u_r = \frac{\nu}{r^n} F(\theta), \quad u_\theta = 0, \tag{1}$$

where ν is the kinematic viscosity.

(a) Find the power n using the conservation of mass in polar coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0.$$

(b) The Navier Stokes equations with the assumptions (1) simplify to

$$\begin{aligned} u_r \frac{\partial u_r}{\partial r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_r}{\partial r} \right) - \frac{u_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} \right), \\ 0 &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}. \end{aligned}$$

Using this set of equations, find an ODE for $F(\theta)$. (Hint: you need to eliminate the pressure terms in the two equations above; you can find $\partial p/\partial\theta$ from the second equation and then take $\partial/\partial\theta$ of the first equation and replace for $\partial p/\partial\theta$.)

(c) Considering the symmetry of the flow and zero velocity at the walls, derive three boundary conditions for $F(\theta)$ which are required to solve the ODE you derived in (b). (You do not need to solve this ODE.)

SECTION B

Q5 The effect of the surface tension can be included in water waves by modifying the pressure boundary condition. The governing equations describing water waves in domain $D_t = \{(x,z) : -\infty < x < \infty, -h < z < \eta(x,t)\}$ then become

$$\begin{aligned} \Delta\phi &= 0 \quad \text{in } -h < z < \eta, \\ \frac{\partial \phi}{\partial t} + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + g\eta - \sigma\frac{\partial^2\eta}{\partial x^2} &= 0 \quad \text{on } z = \eta, \\ \frac{\partial \eta}{\partial t} + \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\eta &= \frac{\partial\phi}{\partial z} \quad \text{on } z = \eta, \\ \frac{\partial\phi}{\partial z} &= 0 \quad \text{on } z = -h. \end{aligned}$$

where σ is a constant.

(a) Linearise these equations to give the modified equations for linear water waves.

(b) Find the dispersion relation for travelling waves.

(c) Show that in the long wavelength limit the phase speed is unaffected by surface tension.

(d) Show that in the short wavelength limit the group velocity is larger than the phase velocity.

(e) Consider a gaussian wave packet travelling in the short wavelength limit. Describe qualitatively how the waves inside the envelope evolve as it propagates.

Q6 Consider the kinetic helicity

$$H = \int_D \boldsymbol{\omega}\cdot\boldsymbol{u}\, dV,$$

of a flow $\boldsymbol{u}$ within a fixed, closed volume D where $\boldsymbol{\omega}\cdot\boldsymbol{n} = 0$ on ∂D.

(a) Show that H is unchanged by the addition of a potential flow, i.e. when $\boldsymbol{u} \to \boldsymbol{u} + \boldsymbol{\nabla}\phi$.

(b) With reference to what H measures, explain why this is the case.

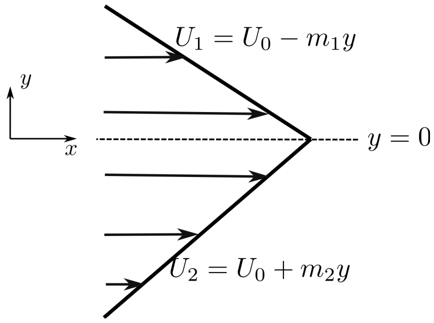
(c) If

$$\boldsymbol{u}(x,y,z) = \exp(-(x-x_0)^2 - y^2)\boldsymbol{e}_z + \exp(-(x+x_0)^2 - z^2)\boldsymbol{e}_y,$$

in an infinite domain, calculate H. You may use that $\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}$.

(d) Sketch the flow streamlines and vortex lines for this flow and use this to explain why H is finite despite the flow itself extending to infinity.

Q7 In this question, we investigate the stability of the edge wave shown in the figure.



The basic velocity vector does not have a vertical component (in the y direction) and its horizontal component (in the x direction) is given as

$$U(x, y) = \begin{cases} U_1 = U_0 - m_1 y & y > 0 \\ U_2 = U_0 + m_2 y & y < 0, \end{cases}$$

where U_0, m_1 and m_2 are constants. To make the stability analysis easier we linearise the vorticity equation (instead of velocity). For instance, if we denoted the variables associated with the top layer with the subscript 1, we derive the linearised vorticity equation as

$$\frac{\partial \xi_1'}{\partial t} + U_1 \frac{\partial \xi_1'}{\partial x} = 0,$$

where $\xi_1' = \partial v_1'/\partial x - \partial u_1'/\partial y$ is the vorticity (perpendicular to the plane) of perturbation fields with u_1' being the velocity perturbation in the x direction and v_1' the velocity perturbation in the y direction. We also use the streamfunction of perturbation velocity such that $u_1' = -\partial \psi_1'/\partial y$ and $v_1' = \partial \psi_1'/\partial x$, and assume the following ansatz for ψ_1' and pressure

$$\psi_1' = \tilde{\psi}_1(y) e^{i(kx - \omega t)}, \quad p_1' = \tilde{p}_1(y) e^{i(kx - \omega t)}.$$

We consider similar ansatz and equations for the variables associated with the bottom layer, i.e. for u_2', v_2', ξ_2', ψ_2' and p_2'.

(a) Rewrite the linearised vorticity equation in terms of $\tilde{\psi}_1$ (and $\tilde{\psi}_2$), and the basic velocities U_1 (and U_2). From these equations find $\tilde{\psi}_1$ and $\tilde{\psi}_2$. You can assume $\tilde{\psi}_1 \to 0$ as $y \to \infty$ and $\tilde{\psi}_2 \to 0$ as $y \to -\infty$.

(b) Show that the continuity of pressure at the edge $y = 0$ leads to

$$\left(U_1 - \frac{\omega}{k}\right) \frac{d\tilde{\psi}_1}{dy} - \tilde{\psi}_1 \frac{dU_1}{dy} = \left(U_2 - \frac{\omega}{k}\right) \frac{d\tilde{\psi}_2}{dy} - \tilde{\psi}_2 \frac{dU_2}{dy}, \quad \text{at } y = 0.$$

(Hint: you can use the linearised horizontal momentum equation).

(c) Argue that the continuity of velocity at the edge results in $\tilde{\psi}_1(0) = \tilde{\psi}_2(0)$.

(d) Using the previous steps, find the dispersion relation for this flow. Based on the dispersion relation argue whether the flow is stable or unstable. Does the stability of this flow depend on m_1 and m_2?

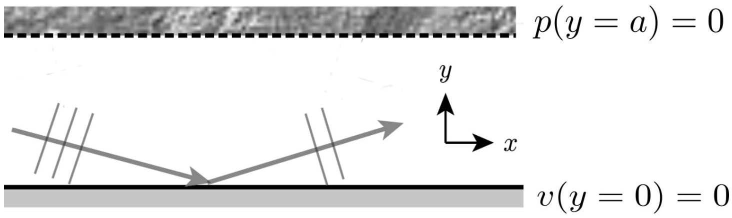
Q8 We consider 2D planar waves in a tube shown in the figure ($(x, y) \in \mathbb{R} \times [0, a]$). At the bottom boundary of this tube, which is a solid material, the vertical velocity is zero. The top boundary is a spongy material that forces the pressure to be zero but allows the flow to have non-zero velocity (this type of boundary damps the sound wave and acts as sound insulation). We start with the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c_0^2 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right),$$

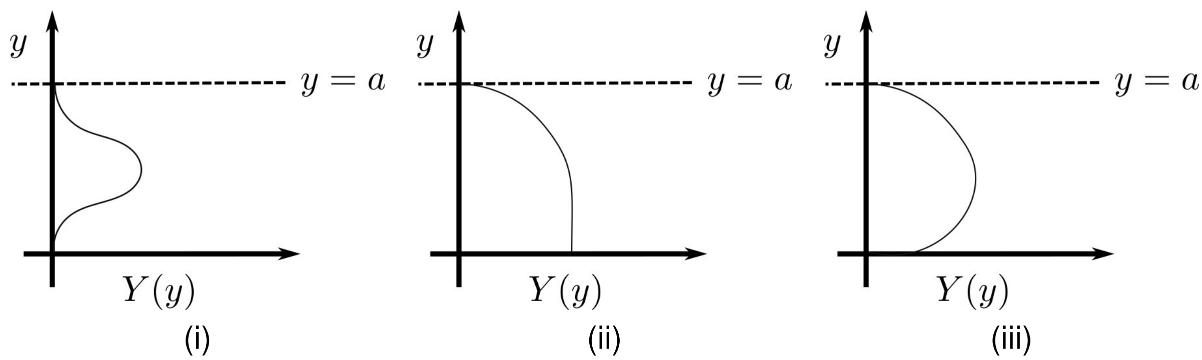
written for the acoustic velocity potential ϕ that satisfies

$$\boldsymbol{u} = (u, v) = \nabla \phi, \quad p = -\rho_0 \frac{\partial \phi}{\partial t}.$$

We look for a solution propagating in the x direction of the form $\phi(x, y, t) = Y(y) \exp(ik_x x - i\omega t)$.



(a) Considering the boundary conditions, discuss whether or not the following examples are valid profiles for $Y(y)$? Explain your reasons for each profile.



(b) Using the wave equation and the correct form of the boundary conditions, find the general solution for this problem.

(c) With $a = \pi$, find $\phi(x, y, t)$ for the wave in this tube that has the initial profile of $\phi(x, y, 0) = \cos^3(y/2)\cos(x)$. (You can use the identity $\cos(3x) = 4\cos^3(x) - 3\cos(x)$)